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ADVANCED NONLINEAR THEORY: LONG-TERM STABILITY AT THE SSC

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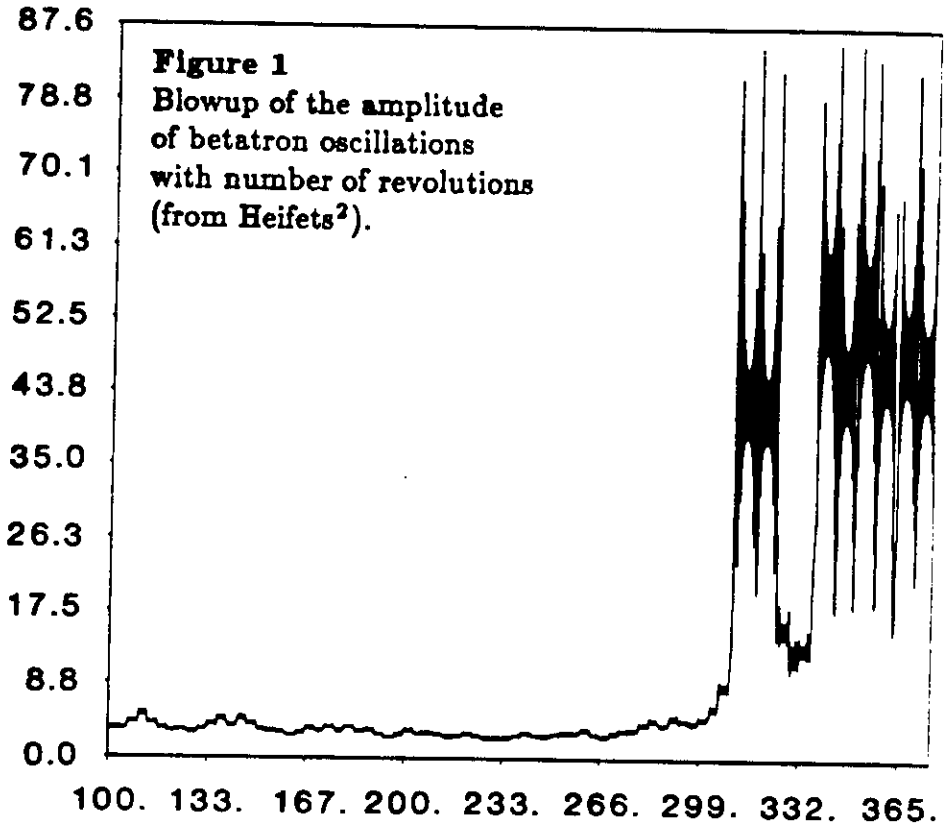


Figure 1
Blowup of the amplitude of betatron oscillations with number of revolutions (from Heifets[2]).

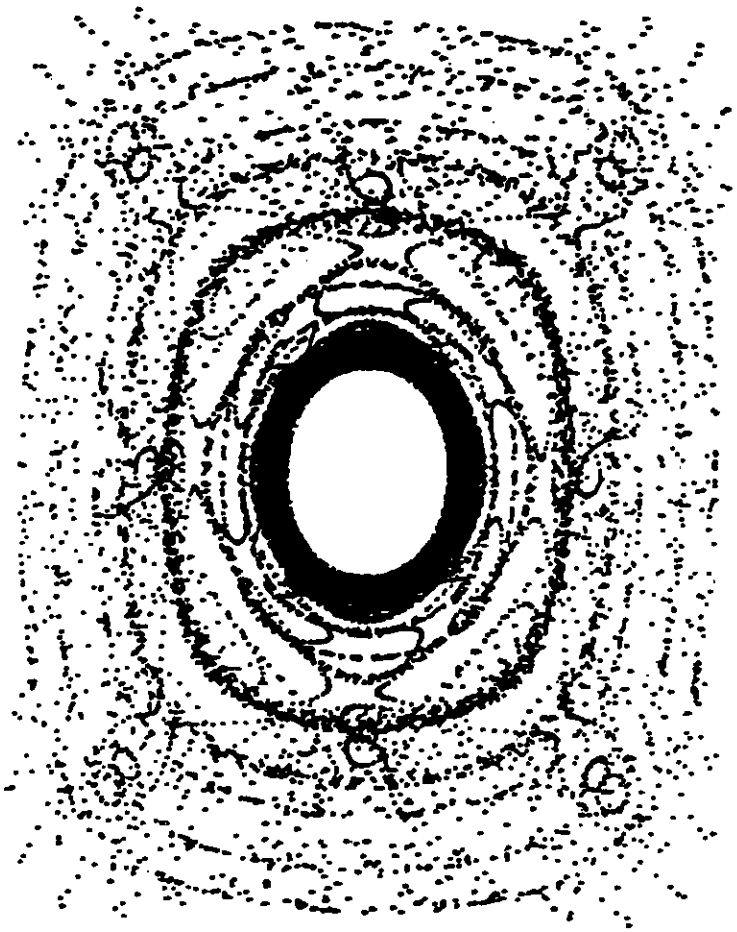


Figure 2
The phase plot for the same case as in Fig. 1.

The theory of alternating gradient accelerators itself started with the classic Courant and Snyder paper.[3] All the major phenomena of one-particle dynamics were already described in Kolomensky and Lebedev's book.[4] Chirikov's criterion of the onset of stochasticity[5] was formulated in 1959. Since then, generations of accelerator physicists have studied rigorously and systematized practically all important effects.[6] Most efforts were related to particular machines and nonlinearities. The main concern was short-term stability, which was studied basically by perturbation theory. Computers made possible the numerical simulation of particle dynamics. Sophisticated numeric codes were developed with emphasis on the simplecticity of the algorithm,[7] and their significance is hard to overestimate. The working accelerators justify and prove the merit of these efforts.

The main progress in understanding long-term stability came, however, from studies of simple models with few degrees of freedom in nonlinear mechanics. Several new phenomena were discovered, such as the scaling law in bifurcation of periodic orbits[8] and the strange attractor in nonlinear dissipative systems.[9] The mechanism of the break-up of Kolmogorov-Arnold-Moser (KAM) invariant tori was recently clarified.[10] A residue criterion, very important for applications, has been formulated.[12] A method of evaluating transit time in the stochastic component has been suggested.[11] The generalization of the results on the three-dimensional time-dependent case, important for accelerators, will be justified soon.[13] It is more difficult to understand, however, how these methods can be applied to real accelerators, where random multipoles, noise, and dissipation are important. As a result of this progress, we are much better prepared now than before to analyze long-term stability.

First of all, KAM theory assures that a small enough and smooth perturbation, although it distorts the trajectories, does not change the character of motion for most of them. This validates the application of perturbation theory to systems with small nonlinearities.

The structure of phase space has a universal character. The regions in phase space with stable, regular motion intermit with stochastic layers, where motion is random. The stochastic component increases with nonlinearity (or amplitude) of the system. When the nonlinearity exceeds some finite threshold, stochastic motion takes place in the whole phase space. The residue criterion or Chirikov's criterion gives the quantative estimation of the onset of global stochasticity.

Below the threshold, where all accelerators operate, there is a drift of particles along the thin stochastic layers. If the system has more than 2 dimensions, then, for conservative systems, particle drift is unbounded. This is Arnold's diffusion, and it causes the universal instability of nonlinear systems. The coupling between stochastic motion in a stochastic layer and other degrees of freedom generates the so-called modulation diffusion in these degrees of freedom.[14]

Noise causes additional streaming along the stochastic layers.[15] All these types of diffusion define the rate of "weak instabilities."[16] Fortunately, the diffusion rate for small nonlinearities is exponentially small, so that accelerators have a chance to work.

Tracking, or numerical calculation of the trajectory of the lattice element by element and turn by turn, is the most powerful method today for the analysis of weak instabilities. However, the method has its limitations, related to limited available CPU time and the generally singular dependence of trajectory behavior on initial conditions.

The goal of this chapter is to review the present situation regarding one-particle dynamics in a collider such as the SSC. Some important aspects are beyond its scope, and many excellent papers are not mentioned, because of limitations of chapter length and of the author's knowledge.

1. HAMILTONIAN; NOTATIONS

For simplicity we assume a plane closed reference orbit of circumference $2\pi R$. The longitudinal coordinate of a particle is the distance along the reference orbit. The particle location specifies the plane, perpendicular to the reference orbit. The coordinate system (x, y) is chosen in this plane with the origin on the closed orbit. The vertical y-axis is perpendicular to the plane of the reference orbit.

In the linear approximation, the Hamiltonian takes the form

$$H_0(x, p_x, y, p_y, s) = \frac{1}{p}\left\{\frac{{P_x}^2}{2} + \frac{{P_y}^2}{2} - \left(\frac{\Delta p}{\rho}\right)X + \left(\frac{p_s}{\rho}\right)\left(\frac{\Delta B}{B_s}\right)X + \frac{p_s}{2\rho}\left(\frac{1}{\rho}X^2 + b_1X^2 - 2a_1XY - b_1Y^2\right)\right\} \tag{1-1}$$

where $P_{x,y} = pd(x, y)/ds$ are canonically conjugated to the coordinates X, Y; the deviation of the momentum p from the equilibrium momentum p_s is $\Delta p = p - p_s$; the distortion of the y-component of the magnetic field $B_y(x, y)$ on the reference orbit is $\Delta B = B_y(x = y = 0) - B_s$; and the radius of curvature of the reference orbit $\rho(s)$ is given by the relation $p_s c = e\rho B_s$. The multipole expansion of the field $B_{x,y}(x, y)$ defines the $(2n+2)$-pole coefficients $b_n(s)$, and skew multipoles $a_n(s)$:

$$B_y + iB_x = B_s \sum_{n=0} (b_n + ia_n)(x + iy)^n .$$

The coefficient b_2 defines sextupole errors, b_3 gives octopole errors, and so on. In CDR b_n are measured in units of 10^{-4} cm^{-n}. We assume the symmetry $B_y(x, -y) = B(x, y)$, which means that all skew multipoles $a_n = 0$.

The terms $n = 0, 1$ are included in the linear Hamiltonian (1–1). The nonlinearities b_n, $n \geq 2$ define the perturbation V:

$$V(X,Y,S) = \frac{p_s}{2\rho} \sum_{n=2} \frac{bn(s)}{n+1} [(X+iY)^{n+1} + c.c.] \,. \qquad (1\text{–}2)$$

For the off-momentum particle the horizontal displacement X is a sum of a periodic closed-orbit function ς (s) and the free betatron oscillations x_b around it:

$$X = \varsigma(s) + x_b(s) \,, \qquad Y = y_b(s) \,;$$

$$P_x = p\varsigma'(s) + p_x(s) \,, \qquad P_y = p_y(s) \,. \qquad (1\text{–}3)$$

The closed-orbit function satisfies the equation

$$\varsigma'' + \frac{1}{\rho}\left(\frac{P_s}{p}\right)\left(b_1 + \frac{1}{\rho}\right)\varsigma = \frac{1}{\rho}\left(\frac{\Delta p}{p}\right) + \frac{1}{\rho}\left(\frac{p_s}{p}\right)(1 - b_0) - \frac{1}{\rho}\left(\frac{p_s}{p}\right) \sum_{n=2} b_n \varsigma^n(s). \quad (1\text{–}4)$$

In the linear approximation in $(\Delta p/p)$ and for $b_0 = 1$, the closed-orbit function is proportional to the dispersion function $\varsigma = (\Delta p/p)\eta_0(s)$. The amplitudes of the betatron oscillations are given by the relations

$$x_b = \sqrt{2\varepsilon_x \beta_x(s)} \cos \psi_x \,, \qquad \frac{p_x}{p} = -\sqrt{2\varepsilon_x/\beta_x} \sin(\psi_x + \alpha_x \cos\psi_x) \qquad (1\text{–}5)$$

and the same in the vertical plane.

Here $\varepsilon = I/p = \bar{x^2}/\beta$, and the action I and angle $\phi = \psi - \theta(s)$ are canonical conjugated variables:

$$\theta(s) = \mu(s) - \nu_0 s/R \,, \qquad \mu(s) = \int_0^s ds/\beta(s) \,, \qquad 2\pi\nu_0 = \mu(2\pi R) \,. \qquad (1\text{–}6)$$

The function μ(s) is called a phase advance. Normalized emittance ε_n, defined in terms of the rms beam size $\sigma^2 = < x^2 >$,

$$\varepsilon_n = \frac{< I >}{mc} = \frac{\gamma\sigma^2}{\beta}$$

is an adiabatic invariant remaining constant during the acceleration. For the SSC, $\varepsilon_n = 1$ μm.

The beta-function $\beta = w^2$ and $\alpha = -(1/2)\beta'(s)$ are given by the periodic solution of the equations

$$w''_{x,y} \pm \frac{1}{\rho}\left(\frac{Ps}{p}\right) n_{x,y} w_{x,y} = {w_{x,y}}^{-3} \tag{1-7}$$

where $$n_y = \sum_{n=1} n b_n \varsigma^{n-1}, \qquad n_x = n_y + \frac{1}{\rho}.$$

The beta-function defined in this way depends on momentum $\Delta p/p$, because of the coefficient (p_s/p), which gives the natural chromaticity, and because of the momentum dependence of the function $\varsigma(s)$, if $b_n \neq 0$. The same is true for the linear tune ν_0, defined by Eq. (1–6).

The Hamiltonian H_0 in the action-angle variables takes the form

$$H_0(I, \phi, s/R) = p\nu_0\varepsilon = p({\nu_x}^0\varepsilon_x + {\nu_y}^0\varepsilon_y). \tag{1-8}$$

The total Hamiltonian is

$$H = H_0 + V, \qquad \frac{dI}{ds/R} = \frac{-\partial H}{\partial \phi}, \qquad \frac{\partial \phi}{ds/R} = \frac{\partial H}{\partial I}. \tag{1-9}$$

The perturbation V is

$$V(I, \phi, s/R) = p\sum g_m(s) {\varepsilon_x}^{M_x} {\varepsilon_y}^{M_y} e^{im\phi} \tag{1-10}$$

where

$$(m) = (m_1; m_2; m_3; m_4), \qquad m\phi = m_x\phi_x + m_y\phi_y,$$

$$M_x = \frac{1}{2}(m_1 + m_2), \qquad M_y = \frac{1}{2}(m_3 + m_4),$$

$$m_x = m_1 - m_2, \qquad m_y = m_3 - m_4.$$

The function $g_m(s)$ is periodic in s:

$$g_m(s) = \left(\frac{p_s}{p}\right)\left(\frac{R}{\rho}\right) N(m) {\beta_x}^{M_x} {\beta_y}^{M_y} e^{im\theta} {b_{k-1}}^{ef},$$

$$N(m) = (-)^{M_y}[1 + (-)^{m3+m4}]\frac{(k-1)!2^{-(1+k/2)}}{m_1!m_2!m_3!m_4!};$$

$${b_k}^{ef} = \sum_{l=k}^{\infty} \frac{l!}{k!(l-k)!} b_l \varsigma^{l-k}; \tag{1-11}$$

all $(m_1 \cdots m_4)$ are integers ≥ 0, $k = 2(M_x + M_y) \geq 3$.
The variables

$$a = \sqrt{\varepsilon}\, e^{i\phi}, \qquad a^+ = \sqrt{\varepsilon}\, e^{-i\phi} \tag{1-12}$$

with the Poisson brackets

$$\{a, a^+\}_{\phi I} = i/p \tag{1-13}$$

sometimes are more convenient than the angle-action variables. In these variables the Hamiltonian $H = H_0 + V$ is

$$H_0(a, a^+, s/R) = p(\nu_x{}^0 a_x{}^+ a_x + \nu_y{}^0 a_y{}^+ a_y), \tag{1-14}$$

$$V(a, a^+ s/R) = p \sum g_m(s) a_x{}^{m1} a_x{}^{+m2} a_y{}^{m3} a_y{}^{+m4}$$

and the Hamiltonian equations are

$$\frac{da}{d(s/R)} = \frac{i}{p}\{H, a\}_{a^+ a} \tag{1-15}$$

complex conjugated for a^+.

If the perturbation V is negligibly small, Eqs. (1–9) give $I_{x,y} = \text{const}$, that is, the famous Courant-Snyder invariant

$$I_x = \frac{P}{2\beta_x}\left[\left(\frac{P_x \beta_x}{p} + \alpha_x X_b\right)^2 + X_b^2\right].$$

The phase-independent term of the perturbation is

$$< V > = p \sum_{M_{x,y}}{}' < g_m(s) > \varepsilon_x^{Mx} \varepsilon_y^{My} \tag{1-16}$$

where the prime means that $m_x = m_y = 0$ in the sum, and $< >$ indicates the average over a period. This term gives the tune shift of the first order in V:

$$\delta\nu_{x,y}{}^{(1)} = \left(\frac{1}{p}\right) \frac{\partial}{\partial \varepsilon_{x,y}} < V >. \tag{1-17}$$

We will usually include $< V >$ in H_0 and use the amplitude-dependent tune

$$\nu_{x,y} = \nu_{x,y}{}^0 + \delta\nu_{x,y}{}^{(1)}.$$

the ratio $< V/H_0 >$ gives a dimensionless parameter that estimates the magnitude of the perturbation. For a multipole b_k the parameter is

$$\lambda_k \sim b_k A^{K-1} \beta (l N_K / 2\pi\rho) \tag{1-18}$$

where $A = \sqrt{\beta\varepsilon}$ is the amplitude of the betatron oscillations, l is the length of a single multipole, and N_k is the number of them per ring.

The parameter λ_k increases with the amplitude. For sextupole errors in the dipole magnets of one unit, $b_2 = 10^{-4}$ cm^{-2}, and with the typical value for the SSC $\beta = 300$ m, the parameter $\lambda_2 \simeq 1$ for the amplitude $A \sim 3$ mm.

The perturbation V depends on the effective multipoles $b_n{}^{ef}$, defined in Eq. (1–11). It means that for the off-momentum particle ($\Delta p/p \neq 0$) higher-order multipoles generate effective low-order multipoles. Say, octupole $b_3 = 1$ unit generates the sextupole-like perturbation with

$$b_2{}^{ef} \sim \frac{3}{2} b_3 \eta_0(s) \left(\frac{\Delta p}{p}\right).$$

Typically for the SSC, $\eta_0(s) = 3$ m, $(\Delta p/p) = 10^{-3}$, this gives $b_2{}^{ef} \sim 0.5$ units. The ratios of the perturbation parameters (1–18)

$$\frac{\lambda_2{}^{ef}}{\lambda_3} \sim \left(\frac{\eta_0}{\sqrt{\varepsilon\beta}}\right)\left(\frac{\Delta p}{p}\right)$$

are of orders of one for the amplitude $\sqrt{\varepsilon\beta} \sim 3$ mm.

Problems

1. Derive the equation of motion for X, Y.

Answer:

$$X'' + \left(\frac{p'}{p}\right)X' + \left(\frac{p_s}{p}\right)\left(\frac{B_y - B_s}{B_s\rho}\right) + \left(\frac{X}{\rho^2}\right) = \left(\frac{1}{\rho}\right)\left(\frac{\Delta p}{p}\right),$$
$$Y'' + \left(\frac{p'}{p}\right)Y' - \frac{B_x}{B_s\rho} = 0\,.$$

2. Find the explicit form of the perturbation (1–2), driven by sextupoles b_2, a_2 and octupoles b_3, a_3.

Answer:

$$V = \frac{P_s}{\rho}\left\{\frac{b_2}{3}(x^3 - 3xy^2) - a_2x^2y + \frac{b_3}{4}(x^4 - 6x^2y^2 + y^4) + a_3(xy^3 - x^3y)\right\}.$$

3. The momentum dependence of the functions $\varsigma, \beta = w^2$ usually can be found by expansion in $\delta = \Delta p/p$:

$$\varsigma = \delta\eta_0 + \delta^2\eta_1 + \dots\,,$$
$$w = w_0 + \delta w_1 + \dots\,,$$

where w_0 satisfies Eq. (1–7) with $b_n = 0$, for $n \geq 2$ and for the on-momentum particle ($p = p_s$). Find the explicit expressions for η_1, w_1.

Answer:

$$\eta_1 = \frac{w_{0x}(s)}{2\sin(\pi\nu_x{}^0)} \int_s^{s+2\pi R} (\frac{ds'}{\rho}) w_{0x}(s')\eta_0(s') \left[\left(b_1 + \frac{1}{\rho} \right) - b_2\eta_0(s') \right]$$
$$\times \cos[\pi\nu_x{}^0 + \mu_x{}^0(s) - \mu_x{}^0(s')] \; ;$$
$$w_{x,y}{}^{(1)} = \pm \frac{w_{x,y}{}^{(0)}}{\sin(2\pi\nu_{x,y}{}^0)} \int_s^{s+2\pi R} \left(\frac{ds'}{4\rho} \right) [b_{x,y} - 2b_2\eta_0(s')]\beta_0(s')$$
$$\times \cos 2[\pi\nu_{x,y}{}^0 + \mu_{x,y}{}^{(s)} - \mu_{x,y}(s')] \; .$$

4. Find the first-order tune shift due to the natural chromaticity and sextupole errors.

Answer:

$$\nu_{x,y}{}^{(1)} = \pm \int_0^{2\pi R} \left(\frac{ds'}{2\pi\rho} \right) \beta_0(s') \left[\frac{1}{2} b_{x,y} - b_2\eta_0(s') \right]$$

where $b_y = b_1(s), \qquad b_x = b_1(s) + 1/\rho \; .$

5. Find the first-order tune shift given by octupole and decapole errors.

Answer:

$$\delta\nu_{x,y} = \frac{\partial}{\partial\varepsilon_{x,y}} (\delta H_n), \text{ where } \delta H_n \text{ is}$$
$$\delta H_n = \sum_{k,l} \frac{(-)^l}{(n+1-2k-2l)!} \left(\frac{\Delta p}{p} \right)^{n+1-2k-2l}$$
$$\times \left(\frac{\varepsilon_x}{2} \right)^k \left(\frac{\varepsilon_y}{2} \right)^l \frac{n!}{(l!k!)^2} < \frac{R}{\rho} \, b_n \beta_x{}^{0k} \beta_y{}^{0l} \eta_0{}^{n+1-2k-2l} > .$$

Here $\beta_{x,y}{}^0$, η_0 are functions calculated for an on-momentum particle $(\Delta p^0/p = 0)$ and $< >$ indicates the average over a period.

6. Find the expression for the dispersion function $\eta_0(s)$ in terms of the beta-function $\beta_0(s)$ (see problem 3).

Answer:

$$\eta_0(s) = \frac{\sqrt{\beta_x{}^0(s)}}{2\sin(\pi\nu_x{}^0)} \int_s^{s+2\pi R} \frac{ds'}{\rho} \cos[\pi\nu_x{}^0 + \mu_x{}^0(s) - \mu_x{}^0(s')] \sqrt{\beta_x{}^0(s')}.$$

2. PERTURBATION THEORY; LIE ALGEBRA METHOD

Here we consider the solution of the Hamiltonian equations in the perturbation theory and the closely related Lie algebra method developed by Dragt. [7] In zero approximation, $H = H_0$, Eq. (1–15) has a solution

$$a = \lambda e^{is/R}, \quad \lambda = \text{constant}. \tag{2–1}$$

With $V \neq 0$, Eq. (2–1) introduces the new canonical variables λ, λ^+, which satisfy the nonlinear equations

$$\frac{d\lambda}{d(s/R)} = i\hat{\mathcal{H}}\lambda; \quad \frac{d\lambda^+}{d(s/R)} = i\hat{\mathcal{H}}\lambda^+ \tag{2–2}$$

where the operator $\hat{\mathcal{H}}$ is

$$\hat{\mathcal{H}}(\lambda, s) = \frac{\partial V(\lambda s)}{\partial \lambda^+}\frac{\partial}{\partial \lambda} - \frac{\partial V(\lambda, s)}{\partial \lambda}\frac{\partial}{\partial \lambda^+}$$

with

$$V(\lambda, s) = \sum g_m(s)\lambda_x{}^{m1}\lambda_x{}^{+m2}\lambda_y{}^{m3}\lambda_y{}^{+m4}e^{im\nu(s/R)}. \tag{2–3}$$

For an arbitrary function $f(\lambda, \lambda^+, s)$,

$$\frac{df}{d(s/R)} = \frac{\partial f}{\partial(s/R)} + i\hat{\mathcal{H}}f.$$

It is convenient to consider the linear equation for the distribution function $F(\lambda, \lambda^+, s)$ rather than the nonlinear Eq. (2–2). A distribution function satisfies the equation $dF/ds = 0$, or Schrodinger-like equation:

$$i\frac{\partial F}{\partial(s/R)} = \hat{\mathcal{H}}F. \tag{2–4}$$

The solution of the equation can be obtained by iteration. It takes the form, well-known in quantum mechanics,

$$F(\lambda, \lambda^+, s) = v_\lambda(s, 0)F^0(\lambda, \lambda^+) \tag{2–5}$$

where $F^0(\lambda, \lambda^+) = F(\lambda, \lambda^+, 0)$ is the distribution function at $s = 0$, and the operator

$$v_\lambda(s, 0) = \hat{T}\exp\left[-i\int_0^s (ds'/R)\hat{\mathcal{H}}(\lambda, s')\right] \tag{2–6}$$

is understood by Taylor expansion of the exponent with time-ordering of the operators:

$$v_\lambda(s,0) = 1 - i\int_0^s \left(\frac{ds'}{R}\right)\widehat{\mathcal{H}} + \frac{(-i)^2}{2}\int_0^s \frac{ds_1}{R}\int_0^{s_1}\frac{ds_2}{R}\widehat{\mathcal{H}}(s_1)\cdot\widehat{\mathcal{H}}(s_2) + \dots . \tag{2-7}$$

For a single trajectory, starting from the point (λ_0, λ_0^+), the distribution function at $s = 0$ is

$$F^0(\lambda,\lambda^+) = \delta(\lambda-\lambda_0)\delta(\lambda^+-\lambda_0^+)$$

and the solution of Eq. (2–2) is given by

$$\lambda(s) = \int d\lambda d\lambda^+ \cdot \lambda \cdot F(\lambda,\lambda^+,s) = \tilde{v}_{\lambda 0}(s,0)\lambda_0 \tag{2-8}$$

with the operator

$$\tilde{v}_{\lambda 0}(s,0) = \tilde{T}\exp\left\{i\int_0^s \frac{ds'}{R}\,\widehat{\mathcal{H}}(\lambda_0,s')\right\}. \tag{2-9}$$

In Eq. (2–8) we integrated by parts and used the operator T, which orders time-dependent operators reversely in time:

$$\tilde{T}\widehat{\mathcal{H}}(s_1)\widehat{\mathcal{H}}(s_2) = \begin{cases} \widehat{\mathcal{H}}(s_1)\widehat{\mathcal{H}}(s_2) & s_1 < s_2 \\ \widehat{\mathcal{H}}(s_2)\widehat{\mathcal{H}}(s_1) & s_2 < s_1 \end{cases}.$$

The same operator $\tilde{v}$ gives the transformation in time of any function $f(\lambda,\lambda^+)$. In particular, for $\varepsilon = \lambda^+\lambda$ we get

$$\varepsilon(s) = \lambda^+(s)\lambda(s) = \tilde{v}_{\lambda 0}(s,0)\lambda_0^+\lambda_0 .$$

The expansion of the operator $\tilde{v}$ in $\widehat{\mathcal{H}}$ can be rewritten as

$$\varepsilon(s) = \varepsilon_0 + i\int_0^s \frac{ds_1}{R}\left\{V(\lambda_0 s_1),\varepsilon_0\right\} - \int_0^s \frac{ds_1}{R}\int_0^{s_1}\frac{ds_2}{R}\left\{V(\lambda_0 s_2),\{V(\lambda_0 s_1),\varepsilon_0\}\right\} + \dots \tag{2-10}$$

i.e. the expansion over the Poisson brackets — the result of the Lie algebra technique.

Equation (2–10) gives the distortion of the Courant-Snyder invariant. In the first order in V it takes the form

$$\Delta\varepsilon_{x,y} = \varepsilon_{x,y}(s) - \varepsilon_{x,y}{}^{0}$$
$$= -i\sum m_{x,y}\varepsilon_x{}^{Mx}\varepsilon_y{}^{My}e^{-im\phi(s)}\int_0^s \frac{ds_1}{R}\, g_m(s_1)e^{im\nu\, s_1/R} \qquad (2\text{–}11)$$

with $\phi(s) = \phi_0 + \nu s/R$.

The parameter of the expansion in Eqs. (2–7) and (2–10) again is the parameter λ_K, Eq. (1–18). Expansion of the operator $\tilde{v}$ in Eq. (2–8) gives the expression for the 4–vector:

$$X_1 = \left[\lambda_x(s), \lambda_x{}^{+}(s), \lambda_y(s), \lambda_y{}^{+}(s)\right]$$

in terms of the initial vector

$$X_i{}^{0} = (\lambda_{0x}, \lambda_{0x}{}^{+}, \lambda_{0y}, \lambda_{0y}{}^{+}) \ :$$

$$X_i = R_{ij}X_j{}^{0} + T_1(i|jk)X_j{}^{0}X_k{}^{0} + T_2X^{03} + T_3X^{04} + \ \ldots \qquad (2\text{–}12)$$

with the matrices $R, T_1, T_2, \ldots$, which depend only on s. (In our variables $\lambda, \lambda^{+}, R_{ij} = 1$.)

The expansion (2–12) is the basic idea of the matrix formalism for paraxial particle optics, developed by Brown.[17]

The operator $\hat{\mathcal{H}}$ is linear in derivatives and polynomial in λ, λ^{+}. The term driven by the multipole $b_k, K \geq 2$ has the generic structure

$$b_K X_0{}^{K}\,\frac{\partial}{\partial X_0}.$$

Thus, in the expansion of the operator $\tilde{v}$ in the series over $\hat{\mathcal{H}}$, the first-order term $\hat{\mathcal{H}}$ gives terms $b_k X_0{}^{k}$, the second-order term $\hat{\mathcal{H}}^2$ gives terms

$$b_k b_l X_0{}^{k+l-1}\,, \qquad (k,l) \geq 2\,, \qquad \text{etc.}$$

This means that, in Eq. (2–12), the matrix T_1 is generated by sextupoles b_2 only, the matrix T_2 is given by octupoles b_3 and second-order sextupoles $b_2{}^{2}$, the matrix T_3 is generated by $b_4; b_2b_3$ and $b_2{}^{3}$, and so on.

By definition, the action of the operator $\hat{\mathcal{H}}(\lambda, s)$ on any function gives the Poisson bracket

$$\hat{\mathcal{H}}(\lambda, s)f = \{V(\lambda, s), f\}_{\lambda\lambda^{+}}.$$

It is easy to show (see problem below) that

$$e^{\widehat{\mathcal{H}}}\Big\{f,g\Big\} = \Big\{e^{\widehat{\mathcal{H}}}f, e^{\widehat{\mathcal{H}}}g\Big\}. \tag{2–13}$$

This immediately proves that the variables $\lambda(s), \lambda^+(s)$, given by Eq. (2–8), are canonical variables for any time s:

$$\{\lambda(s), \lambda^+(s)\}_{\lambda_0,\lambda_0^+} = \{\tilde{v}_{\lambda_0}\lambda_0, \tilde{v}_{\lambda_0}\lambda_0{}^+\} = \tilde{v}\{\lambda_0\lambda_0{}^+\} = 1. \tag{2–14}$$

In other words, the transformation (2–8) is symplectic; for more details, see Dragt.[7]

In practice, the expansion (2–12) has to be truncated. Generally, the truncated expansion is not simplectic, and the variables $\lambda(s), \lambda^+(s)$ are canonical variables only approximately. Errors introduced by truncation are small but are accumulated nevertheless in time. Tracking conducted in this approximation gives physically senseless results, indicating fast growth of the emittances.

By retaining some terms having the same accuracy as the omitted terms, the transformation, remaining approximate, can be made symplectic. For example, if we want to neglect terms $T_2, T_3, \ldots$ in Eq. (2–12), we can omit in Eq. (2–3) all terms except those given by sextupoles, $\mathcal{H} \simeq \mathcal{H}_2 \simeq b_2 X_0{}^2(\partial/\partial X_0)$, and write the transformation in the sympletic form

$$\lambda(s) = \tilde{T}\exp\Big\{i\int_0^s (ds_1/R)\mathcal{H}_2(\lambda_0 s_1)\Big\}\lambda_0 \tag{2–15}$$

rather than

$$\lambda(s) = \Big[1 + i\int_0^s (ds'/R)\mathcal{H}_2\ (\lambda_0, s')\Big]\lambda_0$$

although both expressions have the same accuracy.

Furthermore, with the same accuracy the operators $\widehat{\mathcal{H}}(s)$ of different arguments are permutable, and the operator $\tilde{T}$ can be dropped out. With this simplification, the transformation (2–15) can be found explicitly. Using this method (in the Lie algebra language), Dragt and his colleagues developed the most advanced tracking code MARYLIE, extensively used for the SSC design.

Problem

Prove Eq. (2–13).
Hint: a) Using the Jacobean, identify
$\{c\{a,b\}\} + \{b\{c,a\}\} + \{a\{b,c\}\} = 0$
and prove $\widehat{\mathcal{H}}\{a,b\} = \{\widehat{\mathcal{H}}a, \widehat{\mathcal{H}}b\}$.
b)Prove by induction.

3. CANONICAL TRANSFORMATIONS

The method of canonical transformations is another form of the perturbation theory. The idea is to reduce the magnitude of the perturbation by proper choice of the new variables. The method is well known in classical mechanics[19] and was extensively used for the proof of the KAM theorem.[20] Let us start with the Hamiltonian (1–14) and do the canonical transformation to new variables (α, α^+), so that the new Hamiltonian $\tilde{H}$ does not contain terms of the order of $V(s)$. The generating function

$$\phi(\alpha^+, a, s) = i\alpha^+ a + \psi(\alpha^+, a, s) \tag{3–1}$$

defines

$$a^+ = \alpha^+ - i\,\frac{\partial\psi}{\partial a}\,; \qquad a = \alpha + i\,\frac{\partial\psi}{\partial\alpha^+}\,; \qquad \tilde{H} = H - \frac{\partial\psi}{\partial(s/R)}\,.$$

The explicit relation between (a, a^+) and the new canonical variables (α, α^+) can be found only by iteration. It takes the form

$$\begin{aligned} a &= \alpha + i\,\frac{\partial\psi}{\partial\alpha^+} + i\,\frac{\partial\phi}{\partial\alpha^+} + \frac{1}{2}\left\{\psi, \frac{\partial\psi}{\partial\alpha^+}\right\}_{\alpha\alpha^+} + o(V^3)\,, \\ a^+ &= \alpha^+ - i\,\frac{\partial\psi}{\partial\alpha} - i\,\frac{\partial\phi}{\partial\alpha} - \frac{1}{2}\left\{\psi, \frac{\partial\psi}{\partial\alpha}\right\}_{\alpha\alpha^+} + o(V^3) \end{aligned} \tag{3–2}$$

where $\psi(\alpha, \alpha^+, s)$ and $\phi(\alpha, \alpha^+, s)$ are unknown real functions of the order of V and V^2 respectively.

The new Hamiltonian does not contain the terms of order of V if the function ψ satisfies the equation

$$\frac{\partial\psi}{\partial(s/R)} + i\{\psi, H_0\}_{\alpha\alpha^+} = V(\alpha, \alpha^+, s) - \langle V \rangle \tag{3–3}$$

with the periodic solution of the form

$$\begin{aligned} \psi(\alpha, \alpha^+, s) = {\sum}' C_m {\alpha_x}^{m1} {\alpha_x}^{+m2} {\alpha_y}^{m3} {\alpha_y}^{+m4} \int_s^{s+2\pi R} (ds'/R) g_m(s') \\ \times \exp\{-i(s - s')(m\nu/R)\}\,, \\ C_m = (e^{2\pi i m\nu} - 1)^{-1}\,, \qquad \nu = \partial H_0/\partial(\alpha^+\alpha)\,. \end{aligned} \tag{3–4}$$

In the right side of Eq. (3–3) we subtract the phase-independent part of the perturbation, which has to be included in H_0. Correspondingly, in the sum

Eq. (3–4) the term $m_x = m_y = 0$ is dropped out, and the tunes include the first-order corrections, given by Eq. (1–17).

Similarly, the terms of order V^2 in the new Hamiltonian are canceled out, if the function ϕ satisfies the equation

$$\frac{\partial \phi}{\partial (s/R)} + i\{\phi, H_0\}_{\alpha\alpha^+} = \frac{-i}{2}\{\psi, V\}_{\alpha,\alpha^+} + \frac{i}{2} < \{\psi, V\} > . \qquad (3\text{–}5)$$

Again, the last term in the right side of Eq. (3–5) has to be included in H_0. It gives the tune shift of the second order:

$$\delta\nu_{x,y}{}^{(2)} = \frac{-i}{2} \frac{\partial}{\partial(\alpha^+\alpha)_{x,y}} < \{\psi, V\}_{\alpha\alpha^+} > . \qquad (3\text{–}6)$$

The new Hamiltonian takes the form

$$H(\alpha, \alpha^+, s) = \nu_x{}^0 \alpha_x{}^+ \alpha_x + \nu_y{}^0 \alpha_y{}^+ \alpha_y + < V > - \frac{i}{2} < \{\psi, V\} > + o(V^3). \qquad (3\text{–}7)$$

In principle, the procedure can be repeated to eliminate the terms of order V^3 and so on. The magnitude of the pertubation is reduced very quickly — the method is "superconvergent" — after n iterations the remaining terms are less than V^{2^n}. In practice, however, it is not only complicated technically, but also makes not much sense to proceed further, because the series is actually divergent. We will return to this point later.

With accuracy $o(V^3)$ the Hamiltonian depends only on $e = \alpha^+\alpha$. This means that $e_{x,y}$ is constant in time. Dragt suggested calling this quantity the nonlinear Courant-Snyder invariant (of the first order). The canonical conjugated phases are $\phi_{x,y} = \nu_{x,y}(e) \cdot s/R$, where $\nu_{x,y}$ include the tune shifts of the first and second order.

In 4-d space $(e_x, \phi_x, e_y, \phi_y)$, the trajectory of a particle with constant momentum lies on the 2-d torus e_x =const, e_y =const.

In the old variables a, a^+ the torus is distorted. The linear invariant $\varepsilon = a^+a$ is no longer constant. According to Eq. (3–2), the distortion is given by

$$\Delta\varepsilon = \varepsilon - e = i\{e, \psi_{\alpha\alpha^+}\} + o(V^2) ,$$

which is the same as that given by Eq. (2–11):

$$\Delta\varepsilon_{x,y} = -i \sum m_{x,y} C(m) \varepsilon_x{}^{M_x} \varepsilon_y{}^{M_y} \times \int_s^{s+2\pi R} (ds'/R) g_m(s') e^{-i(s-s')m\nu/R}. \qquad (3\text{–}8)$$

The trajectory in the phase plane (x, p_x) or (y, p_y) winds around the ellipse which it would form in the linear case ε = const, see Fig 3.

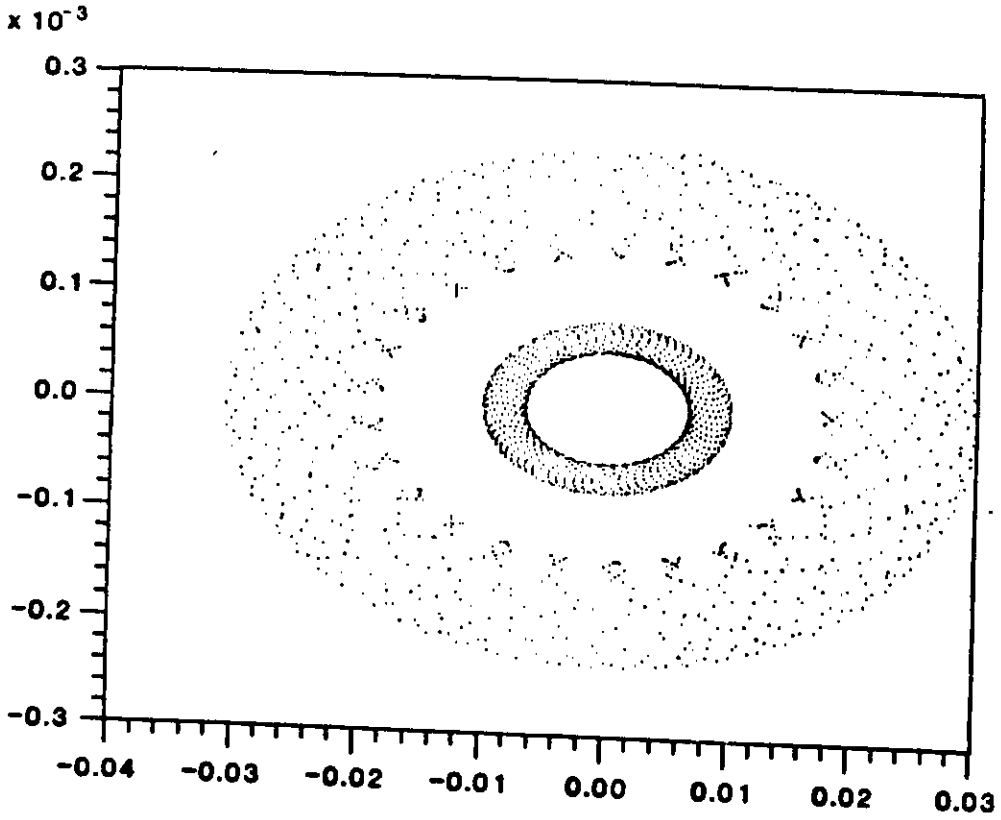


Figure 3
Distortion of two stable trajectories,
obtained with the program MARYLIE.

The formulas (1–17), (3–6), and (3–8) give the tune shifts and distortion function.[21] They can be used to design an achromat, that is, the optical system whose transformation matrix is the identity matrix to a precision of second order, except for the path length, which depends only on $\Delta p/p$. For details, see refs. 17 and 18. To estimate these nonlinear effects, let us assume that multipoles are random with length l and rms value b_k, and there are N_k independent multipoles per ring. The first-order tune shift is given by Eq. (1–17), where

$$< V >= \sum_{k,M_x} N(m) < \beta >^{k/2} (b_{k-1}/\sqrt{N_{\rm tot}})\ \epsilon_x{}^{Mx/2}\epsilon_y{}^{My/2}. \qquad (3\text{–}9)$$

Here $< \beta >$ is the average value of beta-functions and $N_{\rm tot}^{-1} = N_k(l/2\pi R)^2$. If random multipoles are given by the field errors in the dipoles, then $N_{\rm tot} \simeq N_k$. Remember that $k = 2(M_x + M_y), m_x = m_y = 0$. $< V >$ is not zero only for odd multipoles (octupole B_3, dodecapole b_5). The second-order tune shift is given by Eq. (3–6), where

$$\frac{-i}{2} < \{\psi, V\} > = \frac{-\pi}{4} \sum\sum \frac{N(m)N(m') < \beta >^k < b_{k-1}^2 >}{N_{\rm tot} \tan(\pi m\nu) \varepsilon_x^{(k+l'+l)/2} \varepsilon_y^{(k-l'-l)/2}} \times \left[(k+l+l')\frac{m_x}{\varepsilon_x} + (k-l-l')\frac{m_y}{\varepsilon_y}\right]. \tag{3-10}$$

The sum is over k, m_x, m_y, l, l'; the integers $m_\lambda, m'_\lambda, (\lambda, \lambda') = 1,2,3,4$ are given by the relations

$$m_1 = \frac{1}{2}\left(\frac{k}{2} + l + m_x\right), \qquad m'_1 = \frac{1}{2}\left(\frac{k}{2} + l' - m_x\right),$$
$$m_2 = \frac{1}{2}\left(\frac{k}{2} + l - m_x\right), \qquad m'_2 = \frac{1}{2}\left(\frac{k}{2} + l' + m_x\right),$$
$$m_3 = \frac{1}{2}\left(\frac{k}{2} - l + m_y\right), \qquad m'_3 = \frac{1}{2}\left(\frac{k}{2} - l' - m_y\right),$$
$$m_4 = \frac{1}{2}\left(\frac{k}{2} - l - m_y\right), \qquad m'_4 = \frac{1}{2}\left(\frac{k}{2} - l' + m_y\right).$$

The first-order rms distortion, according to Eq. (3–8), is

$$< \Delta\varepsilon_{x,y} > = \left(\frac{\pi m_{x,y}}{\sqrt{N_{\rm tot}}}\right) b_{k-l} N(m) < \beta >^{k/2} \frac{\varepsilon_x^{M_x} \varepsilon_y^{M_y}}{\sin \pi(m\nu)}. \tag{3-11}$$

Comparison of Eqs. (3–9) and (3–11) shows that the tune shift $\delta\nu^{(1)}$ is of order of distortion $< \Delta\varepsilon/\varepsilon >$, and is given by the same parameter λ_k, see Eq. (1–18), which comes up in the perturbation theory. The ratio of the second- and first-order tune shifts, according to the Eqs. (3–9) and (3–10), is given by the same parameter. It also gives the ratio of the second-order distortion $< \Delta\varepsilon/\varepsilon >_2 \sim < \phi/\varepsilon > \simeq < \Delta\varepsilon/\varepsilon >^2$ to the first–order distortion $< \Delta\varepsilon/\varepsilon >_1 \sim < \psi/\varepsilon >$, given by Eq. (3–11).

This could be not true, however, if $m = m_x\nu_x + m_y\nu_y$ is close to an integer, because of the small denominators in Eqs. (3–10) and (3–11) for second-order effects. These "resonances" are considered in the next section. Similar enhancement of the nonlinear effects sometimes takes place also for equal tunes $\nu_x = \nu_y$. Such a "quasi-resonance," driven by the term $g_m(s)$ with $m_1 - m_2 = m_4 - m_3$, exists[22] for octupole term b_3 and for the second-order sextupole effects.[23]

The parameter λ_k and the nonlinear effects increase rapidly with the amplitude of the betatron oscillations. An accelerator can be considered as a system close to a linear system only if $\lambda_k \ll 1$. For the design of the SSC the concept of the "linear aperture" has been formulated. By definition, Eq. (3–1), within the linear aperture the tune shift and the distortion must be smaller than

$$\Delta\nu < 5 \cdot 10^{-3}, \qquad \Delta\varepsilon/\varepsilon < 0.1.$$

For ramdom multipoles, usually the second condition gives more severe restraint.

In CDR these conditions are used to set the tolerances of the multipoles b_k at the amplitude 1 cm. The result is given in Table 4.3–1 of the CDG report.[1]

Given multipoles b_k, the parameters λ_k become big for large amplitudes. Perturbation theory is not applicable for such amplitudes. This is not for lack of a mathematical method, but is a reflection of the drastic change in character of the motion. For large amplitudes the distortion of KAM tori increases until at some finite amplitude the tori completely break up. The trajectories are no longer bounded; instead they start to wander in the phase space. The motion is stochastic, i.e. random, chaotic for such amplitudes. The onset of stochasticity results in diffusion and losses of the particles in an accelerator. This will be discussed later in more detail.

4. RESONANCES: SINGLE RESONANCE; METHOD OF AVERAGING

Perturbation theory gives the answer in terms of integrals [compare Eq.(2–11)] of the form

$$\int_0^s (ds_1/R) g_m(s_1) e^{im\nu(s_1/R)} \ . \tag{4–1}$$

The perturbation $g_m(s)$, being periodic in s with period $2\pi R$, can be expanded in the Fourier series

$$g_m(s) = \sum g_{mk} e^{-ik(s/R)} \ . \tag{4–2}$$

If there is a harmonic for which

$$m\nu = k \ , \qquad (m_x, m_y, k) = \text{integers} \ , \tag{4–3}$$

then the integral (4–1) increases linearly with s, and the perturbation becomes arbitrarily large in time no matter how small the amplitude of the harmonic g_{mk}. The same is true for the method of canonical transformation: the factor $C(m)$ goes to infinity under the condition (4–3).

The increase in nonlinear effects under condition (4–3) is called a nonlinear resonance. On the tune diagram (plane ν_x, ν_y) the condition (4–3) defines a number of straight line resonances, see Fig. 4. How dangerous they are depends, however, on the amplitude of the resonance harmonic. For a smooth perturbation $g_m(s)$ the amplitudes g_{mk} decrease with k. For this reason, the most dangerous are the low-order resonances for which the order of the resonance $|M_x| + |M_y|$ is minimal. A multipole b_n drives only a finite number of

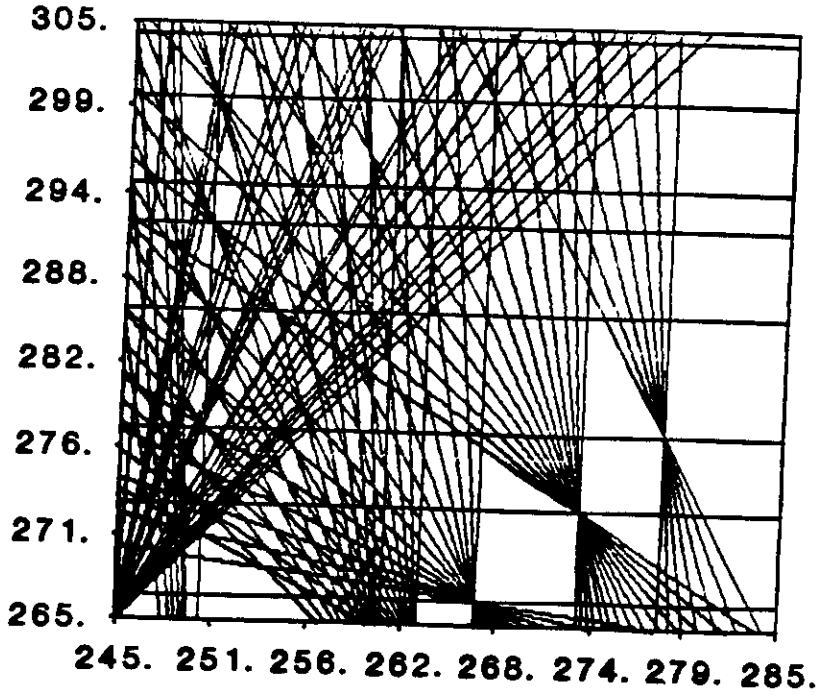


Figure 4
Tune diagram for resonances up to 20th order.

resonances, for which the numeric factor $N(m)$, see Eq. (1–10), is not zero. For example, sextupoles b_2 give only three resonances of the third-order, because $N(m) \neq 0$ only for

$$3\nu_x = k\,, \qquad \nu_k \pm 2\nu_y = k\,, \qquad \nu_x = k\,.$$

The physical reason for the nonlinear resonances is, of course, the coupling between the motion of a particle along the orbit and betatron oscillations. But, unlike the resonances in linear systems, the amplitudes of the nonlinear oscillations do not increase infinitely with time, as could be expected from Eq. (4–1). Crucial here is the amplitude dependence of the tune. The growth of the amplitude in the resonance changes the tune $\nu(\varepsilon)$, so that the resonance condition (4–3) is no longer fulfilled. This stops further growth of the amplitude.

Perturbation theory must be changed to describe this effect. To describe the isolated single resonance $m_x\nu_x + m_y\nu_y = k$, in the first approximation we can drop all harmonics of the perturbation (1–10) except the resonance harmonic. The "resonance Hamiltonian" is

$$H_{\rm res}(I, \phi, s/R) = p[H_0(\varepsilon) + U_{\rm res}(\varepsilon)\cos(m\phi - ks/R)] \qquad (4\text{–}4)$$

where m_x, m_y are fixed by the resonance condition; $U_{\rm res}$ is

$$U_{\rm res}(\varepsilon_x,\varepsilon_y)=2\sum_{M_x M_y}|g_{mk}|\varepsilon_x{}^{Mx}\varepsilon_y{}^{My}\,;$$

and H_0 includes the corrections (3–9) and (3–10).

It is convenient to introduce new pairs of the canonical variables (ψ,A) and (ϕ,B):

$$\psi=m_x\phi_x+m_y\phi_y-ks/R\,,\qquad \phi=m_y\phi_x-m_x\phi_y\,,$$

$$J=(m_xI_x+m_yI_y)/(m_x^2+m_y^2)\,,$$

$$B=(m_yI_x-m_xI_y)/(m_x^2+m_y^2)\,,$$

$$\{\psi,J\}=1,\qquad \{\phi,B\}=1,\qquad \{\psi,B\}=\{\phi,J\}=0 \tag{4–5}$$

The old I_x,I_y are related to J,B by

$$I_x=m_xJ+m_yB\,,\qquad I_y=m_yJ-m_xB\,.$$

The new Hamiltonian is time independent:

$$H_{\rm res}(J,\psi,B,\phi)=p[H_0-kJ+U_{\rm res}(J,B)\cos\psi]\,, \tag{4–6}$$

and is independent of ϕ. This means that B =const.

The resonance Hamiltonian is the Hamiltonian of a one-dimensional pendulum. The Hamiltonian equations are

$$\frac{d\psi}{d(s/R)}=\frac{\partial H_{\rm res}}{\partial J},\qquad \frac{dJ}{d(s/R)}=\frac{-\partial H_{\rm res}}{\partial\psi}. \tag{4–7}$$

The equilibrium $(\psi_{\rm res},J_{\rm res})$ is given by the conditions

$$\sin\psi_{\rm res}=0\,,\qquad \nu(J_{\rm res})=(\partial H_0/\partial J)_{\rm res}=k\,. \tag{4–8}$$

The motion around the stable fixed point $(\psi_{\rm res},J_{\rm res})$ for small $x=\psi-\psi_{\rm res}$ is given by

$$H_{\rm res}\;\simeq\;p\left(\frac{\nu'(J-J_{\rm res})^2}{2}+\frac{|U_{\rm res}|x^2}{2}\right), \tag{4–9}$$

$$\text{where}\qquad \nu'=\left(\frac{\partial^2H_0}{\partial J^2}\right)_{J_{\rm res}}=m_x{}^2\,\frac{\partial\nu_x}{\partial I_x}+m_y{}^2\,\frac{\partial\nu_y}{\partial I_y}+2m_xm_y\frac{\partial\nu_x}{\partial I_y}\,. \tag{4–10}$$

The Hamiltonian (4–9) describes the linear oscillations

$$\psi = \psi_0 \cos(\Omega s/R) , \qquad J - J_{\rm res} = \Delta J \sin(\Omega/R)$$

with the phase frequency

$$\Omega = \sqrt{\nu' |U_{\rm res}|} .$$

The time dependence of the betatron phases $\phi_{x,y}$ is linear plus phase oscillations,

$$\phi_{s,y} \sim \nu_{\rm res} \cdot s/R + \psi_0 \cos(\Omega s/R) .$$

For larger amplitudes the motion is not linear, and it can be described as a superposition of the oscillations with frequencies $n\Omega$, $n = 1, 2, \ldots$.

If the amplitude exceeds the maximum, the motion becomes unbounded: the phase ψ increases with time. The maximum can be found easily if the resonance Hamiltonian is approximated by the expression

$$H_{\rm res} = p[\nu'(J - J_r)^2/2 + U_{\rm res}(J_{\rm res}, B_{\rm res}) \cos\psi]. \qquad (4\text{–}11)$$

The reason for doing this is that $(J - J_r)$ remains small even when ψ is of order 1. In Eq. (4–11) the second term describes the potential well with depth $2U_{\rm res}$. The maximum amplitudes of the oscillators are

$$\Delta J = (J - J_r)_{\rm max} = \pm\sqrt{4|U_{\rm res}|/\nu'}. \qquad (4\text{–}12)$$

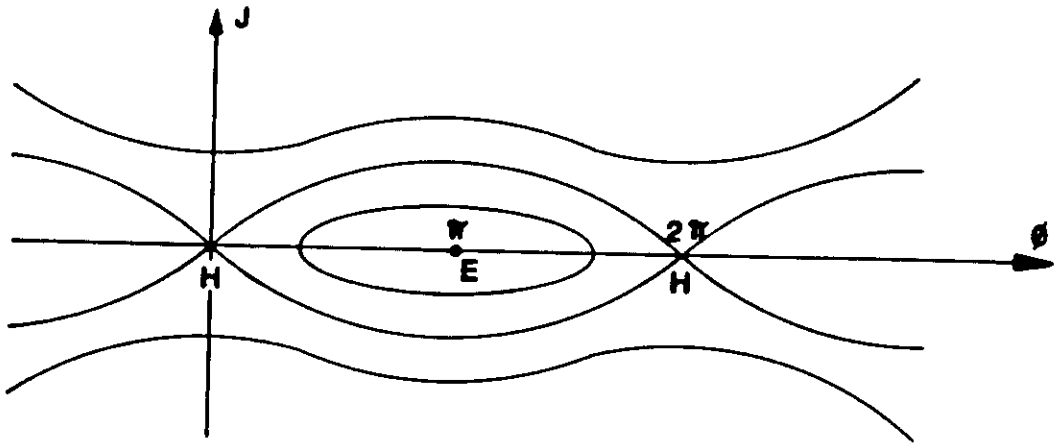
If the nonlinearity $\nu' \to 0$, then $\Delta J \to \infty$, as it would in the linear case. The trajectory with the maximum amplitude (4–12) is called the separatrix (sometimes the same name is used for the area of the phase space inside the separatrix). The motion inside the separatrix corresponds to oscillations; outside it corresponds to the rotation of a pendulum.

The separatrix itself connects two fixed but unstable points, see Fig 5. The motion on the separtrix is aperiodic: it takes infinite time to go from one unstable fixed point to another.

The motion within the separatrix corresponds to the tune in the range $\nu - \nu_{\rm res} = \pm\nu'\Delta J$. The width of the resonance line on the tune diagram is

$$\Delta\nu_{\rm sep} = 4\Omega/\sqrt{m_x{}^2 + m_y{}^2}. \qquad (4\text{–}13)$$

The other resonances of the same harmonic (m_x, m_y) of the perturbation are separated by $\delta\nu \sim 1/m_{x,y}$. The approximation of isolated resonances is valid, therefore, if $4\Omega \ll 1$. Otherwise the resonances cannot be considered separately. Chirikov[5] conjectured that in the case of overlapping resonances $\Delta\nu_{\rm sep} \geq \delta\nu$ a particle starts to jump from one sepratrix to another, walking randomly in the phase space. The Chirikov overlapping criterion has since been proven in a number of numerical simulations. It gives a simple and surprisingly good quantative criterion for the onset of stochasticity.



E - elliptic point
H - hyperbolic points

Figure 5
The separatrix of a nonlinear oscillator.

The higher orders of perturbation theory give new resonances with amplitudes of order $b_2{}^2, b_2{}^3$, etc. They are again given by the condition (4–3). The number of resonances of nth order in b_k (with amplitudes proportional to $b_k{}^n$) increases as n^2. Their width is proportional to the square root of the amplitude, which decreases with n as $\lambda_k{}^{n/2}$, where λ_k is the parameter of perturbation theory (1–18) (see also the discussion in Section 3). The total width of the nth generation resonances increases therefore as

$$\Delta\nu_{\text{tot}} \sim n^2\lambda_k{}^{n/2} . \tag{4–14}$$

Thus, if $\lambda_k \ll 1$, the width of the higher-order resonances decreases exponentially, faster than the number of resonances increases. The total width of all resonances is of the order of that of low-order resonances of the first generation.

On the other hand, for $\lambda_k > 1$, the resonances unavoidably overlap, and the motion, according to the Chirikov criterion, is chaotic.

The parameter λ_k is of the order of magnitude of the first-order tune shift and distortion, as noted in Section 3. In view of this, the concept of the linear aperture (3–12), where $\lambda_k \ll 1$, is very important. A particle has a chance to live long only within the linear aperture. Outside it, where $\lambda > 1$ and motion is chaotic, it gets lost very fast.

Considering the single resonance, we neglected all terms of the perturbation (1–10) except the resonance harmonics. They can be taken into account now by the usual perturbation theory. The perturbation has harmonics with frequencies

$$k + m_x\nu_x + m_y\nu_y$$

with integers k, m_x, m_y. We can expect new resonances if the condition

$$n\Omega = k + m_x\nu_x + m_y\nu_y \tag{4-15}$$

is satisfied. Because $\Omega \ll 1$ [see Eq. (4-14)], the resonances (4-15) are driven by the nth harmonics of the potential $U_{\rm res}\cos\psi$ with $n \gg 1$. The amplitude of the harmonics can be estimated from the expansion

$$U_{\rm res}\cos\psi = U_{\rm res}\sum_n (-)^n\psi^n/n!\,.$$

Thus the amplitude of the nth harmonic with frequency $n\Omega$ is of order

$$U_{\rm res}{\psi_0}^n/n! \;\sim\; U_{\rm res}e^{-n\ln(n/\psi_0)}$$

and can give resonances having a width exponentially small compared with the width of the separatrix of the primary resonances. Nevertheless, they play a very important role, generating the so-called stochastic layer in the vicinity of the separatrix. A more detailed discussion is given in the next section.

We mention here that the higher-order resonances can be considered rigorously by Bogolubov's method of averaging. According to this method[4] we should start with Eq. (2-2)

$$\frac{-id\alpha}{d(s/R)} = \frac{\partial V(\alpha,\alpha^+,s)}{\partial\alpha^+} \tag{4-16}$$

where the potential V (2-3) explicitly depends on s. The variable α should be split into a slow (and generally large) part $\bar\alpha$ and a fast oscillation part $\tilde\alpha$: $\alpha = \bar\alpha + \tilde\alpha$. The potential now can be expanded in $\tilde\alpha$:

$$\frac{\partial V}{\partial\alpha} = \frac{\partial V(\bar\alpha,S)}{\partial\bar\alpha^+} + \tilde\alpha\frac{\partial^2 V(\bar\alpha,s)}{\partial\bar\alpha^+\cdot\partial\bar\alpha} + \tilde\alpha^+\frac{\partial^2 V(\bar\alpha,s)}{\partial\bar\alpha}. \tag{4-17}$$

The equation that is first-order for $\tilde\alpha$ is

$$\frac{-id\tilde\alpha}{d(s/R)} = \left(\widetilde{\frac{\partial V}{\partial\bar\alpha^+}}\right) \equiv \frac{\partial V(\bar\alpha,s)}{\partial\bar\alpha^+} - <\frac{\partial V(\bar\alpha,s)}{\partial\bar\alpha^+}>. \tag{4-18}$$

The last term is averaged over the explicit dependence on s, so that the whole right side of Eq. (4-18) oscillates rapidly. The solution of Eq. (4-18)

$$\tilde\alpha = i\int_0^s ds'\left(\widetilde{\frac{\partial V}{\partial\bar\alpha^+}}\right)$$

gives the expression $\partial V/\partial\alpha$, Eq. (4-17), which depends only on the slow variable $\bar\alpha, \bar\alpha^+$. Its average determines the equation for the slow variable $\bar\alpha$:

$$\frac{-id\bar{\alpha}}{d(s/R)} = < \frac{\partial V(\bar{\alpha}, s)}{\partial \bar{\alpha}^+} > - < \int^s ds' \left\{ \frac{\partial V(s)}{\partial \bar{\alpha}^+}, \tilde{V}(s') \right\}_{\bar{\alpha}, \bar{\alpha}^+} > . \qquad (4\text{–}19)$$

In the first approximation in V, the answer is equivalent to averaging Eq. (4–16) over the explicit dependence on s. If there are resonance terms in V with $\Delta m\nu \simeq k$, they have to be considered as slow functions of s and be retained in averaging. This was used above in considering single resonances.

Problem

Find first-order resonances driven by octupole and decapole nonlinearities.

Answer: The decapole resonances are

$$5\nu_x = k\,, \qquad 3\nu_x \pm 2\nu_y = k\,; \qquad \nu_x \pm 4\nu_y = k$$

plus integer and $1/3$ resonances; the octupole resonances are

$$4\nu_x = k\,, \qquad 2(\nu_x \pm \nu_y) = k$$

plus integer and $1/2$ resonances.

5. INTERACTION OF THE RESONANCES

Considering a single resonance in Section 4, we neglected all fast oscillating terms, in accordance with the method of averaging. Let us study the approximation more carefully. We start with consideration[28] of only one oscillating term:

$$H = H_{\rm res}(j, \psi) + U_{\rm res}(j, B)\cos(\psi - s/R)$$

where $H_{\rm res}$ is defined by Eq. (4–6).

Since we are considering effects that are important only in the close vicinity of the separatrix, we can simplify the Hamiltonian to the form

$$H(j, \psi, s/R) = H_{\rm res} + U_0 \cos(\psi - s/R) \qquad (5\text{–}1)$$

with

$$H_{\rm res} = \frac{\nu'(j - j_0)^2}{2} + U_0 \cos\psi \qquad (5\text{–}2)$$

where $U_0 = U_{\rm res}(j_0, B)$ and $\nu' = \nu'(j_0, B)$, see Eq. (4–9). We assume $U_0 > 0$. The resonance Hamiltonian describes the phase oscillations. For small amplitudes ($\psi \simeq \pi$) the frequency of the oscillations is

$$\Omega = \sqrt{\nu' U_0}\,.$$

If H_{res} approaches the energy on the separatrix $H_{max} = U_0$, the period of the oscillation depends on the amplitude. For small $h = 1 - H_{res}/U_0$ it can be found from Eq. (5–2):

$$(\psi'/2\Omega)^2 = \sin^2(\psi/2) - \sin^2(\psi_m/2) \tag{5–3}$$

where the reflection point ψ_m is defined as

$$\sin^2(\psi_m/2) = h/2 \tag{5–4}$$

and we used the Hamiltonian equation, $\psi' = \nu'(j - j_0)$. Equation (5–3) gives the period for $h \ll 1$,

$$T = \frac{2}{\Omega} \ln \frac{32}{h}, \tag{5–5}$$

and the dependence $\psi(s)$ for $\psi \gg \psi_m$,

$$\frac{(s - s_m)}{R} = \frac{1}{\Omega}\left[\ln \tan\left(\frac{\psi}{4}\right) - \ln \tan\left(\frac{\psi_m}{4}\right)\right]. \tag{5–6}$$

Here $\psi(s_m) = \psi_m$. If the perturbation is included in Eq. (5–1), H_{res} is no longer constant:

$$dH_{res}/d(s/R) = \{H_0, H\}_{\psi,j} = U_0\psi'(s)\sin(\psi - s/R). \tag{5–7}$$

The variation of the energy H_{res} per half period is

$$\Delta h = -\int_{\psi m}^{2\pi - \psi m} d\psi \sin(\psi - s/R)\,.$$

The integral is exponentially small because $\psi' \sim \Omega \ll 1$. It can be extimated by the saddlepoint method, using Eq. (5–6). The contour of the integration must be shifted to go through the saddlepoints $\psi_0^{\pm} = \pi \pm 2i\ln(1/\Omega)$. After some calculation, we get

$$\Delta h = -\left(\frac{8\pi}{\Omega^2}\right) e^{-\pi/s\Omega} \sin\left(\frac{s_m}{R} + \frac{1}{\Omega}\ln\frac{4}{\psi_m}\right). \tag{5–8}$$

Here we used the factor 8π, given by more accurate calculations,[19] instead of $2\sqrt{\pi}e^2 \simeq 8\pi$, which is given by the saddlepoint method. The argument

$$X(h) = \frac{s_m}{R} + \frac{1}{\Omega}\ln\frac{4}{\psi_m} = \frac{s_m}{R} + \frac{1}{4}T(h)$$

is the time at which the oscillator passes the equilibrium phase $\psi = \pi$. The next crossing of the equilibrium occurs at

$$\bar{X} = X + \frac{1}{2}T(h + \Delta h) = X + \frac{1}{\Omega}\ln\frac{32}{\bar{h}} \tag{5–9}$$

where

$$\bar{h} = h + \Delta h\,. \tag{5–10}$$

The system of Eqs. (5–9) and (5–10) gives a map — a transformation $(h, X) \to (\bar{h}, \bar{X})$ in discrete time.

It is worth noting that the variation of the energy h is gained in a narrow interval around the saddlepoint, whereas the phase X increases during the time between successive crossings of the saddlepoint. It is a rather typical situation, which allows us to reduce differential equations of motion to a mapping, approximating a force with a δ-functional kick.

The system (5–9) and (5–10) has fixed points, h_n, which are mapped to themselves. They are given by the condition

$$\frac{1}{\Omega}\ln\frac{32}{h_n} = 2\pi n\,. \tag{5–11}$$

The motion near the fixed points can be described by the linearized equations

$$\bar{X} = X + 2\pi\bar{I}(\mathrm{mod}2\pi)\,, \qquad \bar{I} = I - \frac{K}{2\pi}\sin X \tag{2–12}$$

where

$$2\pi\Omega I = (1 - h/h_n) \ll 1 \tag{5–13}$$

and

$$K = (8\pi/h_n\Omega^3)e^{-\pi/2\Omega}. \tag{5–14}$$

The system (5–12) is called a standard map. It is equivalent to the system with the Hamiltonian

$$H(I, X, s/R) = \frac{I^2}{2} - \frac{K}{(2\pi)^2}\cos X\sum_{-\infty}^{\infty} e^{ik(s/R)}.$$

The Hamiltonian gives resonances for all integers $I_n = k$ having the width $\Delta I = (2/\pi)\sqrt{K}$. The overlap of the resonances results in the stochastic behavior of the system. Numeric simulations predict the transition to chaos for $K > 1$. From Eqs. (5–11) and (5–14) it is easy to see that the motion is stochastic in the vicinity of all fixed points with sufficiently large n:

$$h_n = 32e^{-2\pi n\Omega} < (8\pi/\Omega^3)e^{-\pi/2\Omega} = h_{\max}\,. \tag{5–15}$$

They are all in the exponentially thin layer close to the separatrix of the primary resonance. In many-dimentional systems these areas are connected, giving a web of thin shochastic layers along which a particle can drift away from the initial location. This Arnold diffusion is the cause of the universal instability of nonlinear systems.

The rate of diffusion for $\Omega \ll 1$ is exponentially small. It can be estimated from Eq. (5–8). The variation of the tune per half period is

$$\Delta\nu_{T/2} = \nu'\Delta j = \sqrt{(\Omega^2/2)}\Delta h = -(8\pi/\Omega\sqrt{2})e^{-\pi/2\Omega}\sin X\ .$$

With random phases X, it gives the diffusion rate,

$$D_\nu = \frac{1}{2}\frac{\partial}{\partial(s/R)} < {\Delta\nu_{T/2}}^2 >= \left(\frac{16\pi^2}{\Omega^2 T}\right)e^{-\pi/\Omega}.$$

The period T depends on h, and the diffusion rate decreases close to the separatrix. The average over the interval $0 < h < h_{\max}$, with $h_{\max}$ given by (5–15), gives $< 1/T >= \Omega^2/\pi$, so that

$$\overline{D_\nu} = 16\pi e^{-\pi/\Omega}. \qquad (5\text{–}16)$$

The numeric factor must be considered a crude estimation, and it depends on the detailed structure of the shochastic layer.

The rate (5–16) gives the rms diffusion $\overline{\Delta\nu} \lesssim 0.01$ during 10^8 turns, if $\Omega < 0.09$. This constrains the width of the primary resonance of the mth order, $\Delta\nu_{\text{sep}} = 4\Omega/m$.

The residue criterion (see next section), however, gives even tougher restraints on $\Omega, m\Omega < 1/6$, so that Arnold diffusion does not give rise to additional problems.

Up to now, we have considered the effect of the single closest harmonics of the perturbation. Similarly, other harmonics, $U_0\cos[\psi - k(s/R)]$, $k > 1$, can be considered. They give the same result as that given by Eq. (5–16), but with Ω/k instead of Ω. This means that they are negligibly small, and only interaction of the neighboring resonances is important.

6. STRUCTURE OF THE PHASE SPACE; THE RESIDUE CRITERION

This section is based on review articles by MacKay,[11] Hellman,[24] Hellman and Kheifets,[25] and Ruth,[26] where other references can be found.

The ideal accelerator is a linear conservative system. Such a system is always integrable: for a linear system with n degrees of freedom there are n integrals of motion. The motion can be described as superposition of the normal modes. In the action-angle variables the action I for each mode is the integral

of motion. In $(2n-1)$-dimensional phase space of constant energy $(I_k, \phi_K, k = 1, \ldots, n)$, these variables specify the n-dimensional surface, which is called the n-d torus. The tune can be defined as the change of angle ϕ_k per period, $\nu_k = 1/2\pi\ [\phi_k(s + 2\pi R) - \phi_k(s)]$.

For a nonlinear but integrable conservative system in a proper variables, the Hamiltonian $H(I_k)$ does not depend on phases ϕ_k so that $I_k = \text{const}$. Since the motion is again confined to the surface of the n-d torus, it is bounded in time. The tunes $\nu_k = \partial H/\partial I_k$ depend, however, on the amplitudes. The trajectories with irrational tunes describe so-called quasi-periodic motion. They cover the surface of the torus densely. The trajectories with rational commensurate tunes are closed. The motion can be described as n uncoupled nonlinear oscillations. Two-d phase space for each oscillator looks like that in Fig 4. There are elliptic (stable) fixes points and unstable (hyperbolic) points. Two hyperbolic points are connected by the separatrix of finite length.

Most real nonlinear systems are nonintegrable, and integrable Hamiltonians are rare. A single FODO cell containing a sextupole is the system well-known in nonlinear mechanics as the Hénon-Heiles system, and it is nonintegrable.

All perturbation theories imply that the infinite series of the theory can be truncated. This means that they approximate the nonintegrable system with an integrable one. For example, after n canonical transformations, omitting the small remaining terms, the Hamiltonian is reduced to the form $H = \nu I$, which is obviously integrable. Because the initial Hamilonian is nonintegrable, the series of the perturbation theory must be divergent. They describe the system approximately, only for a finite period of time.

Nevertheless, according to the KAM theorem, under a small and smooth nonlinear perturbation, n-d tori with sufficiently irrational tunes are distorted but preserved. Most of the trajectories lie on the invariant tori, but a finite fraction of trajectories are stochastic and lie between KAM tori. This intermittent structure of the phase space is typical for nonlinear systems and looks the same under any magnigication, see Fig 6. The character of a trajectory has a singular dependence on the initial conditions; small variations can make a regular trajectory chaotic.

Two processes generate chaotic trajectories. One of them gives a stochastic layer in the close vicinity of a separatrix, which is caused by overlapping of the secondary resonances, driven by the interaction of the primary resonances. More detailed consideration shows that the separatrix makes an infinite number of loops, going back and forth inside the stochastic layers, so that the length of the separatrix is infinite, Fig 7. For small perturbation, most of the trajectories inside the separatrix remain stable. Thus the separatrices confine the islands of regular motion, surrounded by stochastic layers near the separatrices. If the nonlinearity increases, the stochastic layers become wider and the islands of stability shrink. Generally, all stochastic layers are connected in one stochastic component of the phase space. For any two points in the stochastic

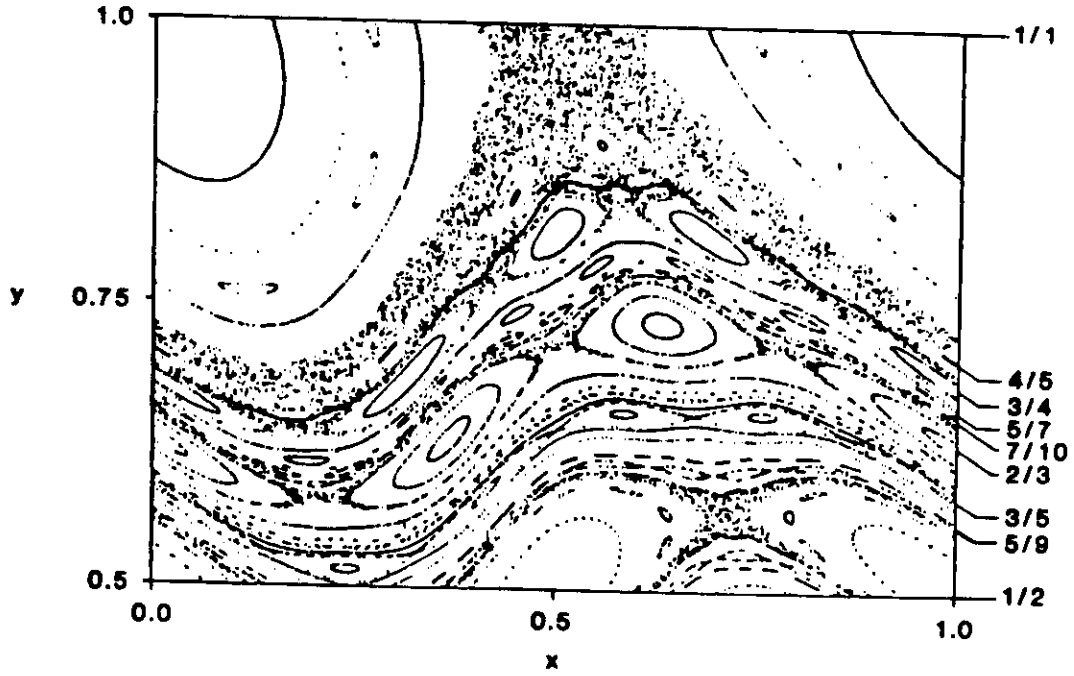


Figure 6

Some orbits of the standard map at $K = 0.9$ (from Mackay[11]).

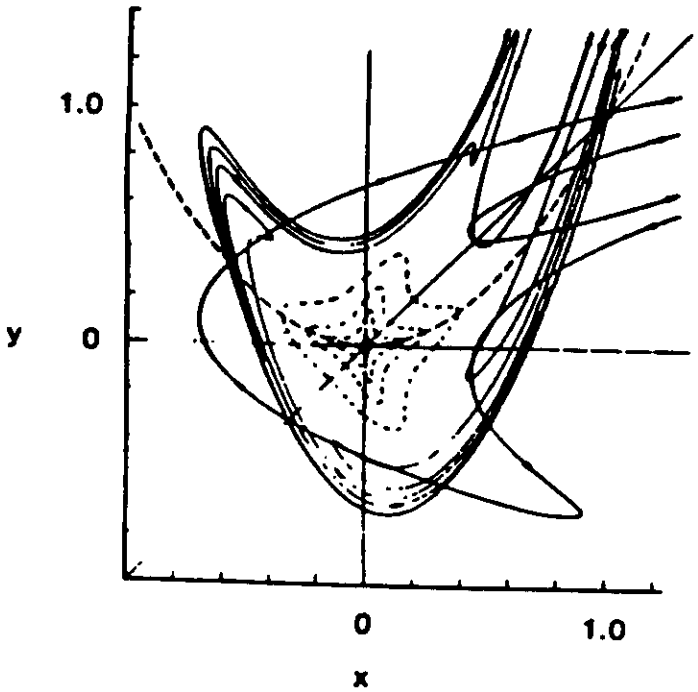


Figure 7

The trajectory in the vicinity of the separatrix for a nonintegrable map, $\bar{x} = y,\ \bar{y} = -x + 2y^2$ (from McMillan[35]).

component, there is a trajectory that connects arbitrary small vicinities of the two points. If the trajectory is stochastic, the particle can drift away arbitrarily far from the initial conditions. Motion is unbounded even if stochastic layers are thin. It is Arnold diffusion. The exception is the systems with two degrees of freedom. For them, 2-d invariant KAM tori divide the 3-d space of constant energy so that chaotic trajectories remain confined between the invariant tori, and the motion is bounded.

For larger nonlinearities, the elliptic periodic orbits, which give fixed points in the centers of the islands, become unstable, splitting into two orbits, one stable and one unstable. These period-doubling bifurcations occur for some sequences of the critical values of the nonlinearity. This universal "Feigenbaum sequence" of the critical values is convergent, so that the process generates an infinite number of hyperbolic and elliptic orbits as the nonlinearity approaches some finite limit. As a result, a single island of stability breaks up into smaller and smaller islands, separated by the chaotic layers. In the limit for the finite critical nonlinearity, the process ends up giving global stochasticity.

The motion in the stochastic component is ergodic (this means that the trajectory covers all stochastic components of the phase space densely); there is mixing (i.e. the exponential decay of initial correlatious), and close trajectories diverge exponentially in time; motion is random (can be considered as the Markov process), etc.

The stochastic component is not uniform. A particle spends a lot of time penetrating barriers that are remnants of the broken invariant tori. The transit time between two parts of the stochastic component scales is

$$\tau \sim (K - K_{\rm cr})^{-\lambda}; \qquad \lambda \sim 3.0$$

for a nonlinearity K larger than its critical value $K_{\rm cr}$.[11,27]

Most of the rigorous results in nonlinear mechanics have been obtained for systems with two degree of freedom. It is convinient to consider the intersection of a trajectory with a 2-d plane (Poincaré surface of section). The angle (divided by 2π) between two successive points of intersection of the trajectory with the surface of section gives the winding number (tune) of the trajectory.

The relation between successive points (I, θ) at $S_n = 2\pi R n$, and $(\bar{I}, \bar{\theta})$ at $S_{n+1} = 2\pi R(n+1)$ describes the dynamics of the system at a discrete time $\sigma_n = s \bmod 2\pi R$, and is very convenient for numeric simulation. This is called mapping. For example, the motion of a particle in a linear lattice containing a single multipole can be described in the thin-lens appoximation by the model Hamiltonian:

$$H(I, \phi, s/R) = (\nu'/2)(I - I_0)^2 - V_0 \cos\phi \sum_n \delta(\sigma - n)\,. \qquad (6\text{–}1)$$

The Hamiltonian equations takes the form

$$\frac{d\psi}{d\sigma} = j\ ; \qquad \frac{dj}{d\sigma} = -\frac{K}{2\pi m}\sin 2\pi m\psi \sum_n \delta(\sigma - n) \tag{6–2}$$

where

$$\psi = \phi/2\pi\ ; \qquad j = \nu'(I - I_0)\ , \qquad \text{and} \qquad K = 4\pi^2 m^2 \nu' V_0\ . \tag{6–3}$$

We assume in the following that $K > 0$. Equations (6–2) are equivalent to the map

$$\bar{\psi} = \psi + j \bmod(1)\ , \qquad \bar{j} = j - (K/2\pi m)\sin 2\pi m\bar{\psi}\ . \tag{6–4}$$

For $m = 1$ this is the "standard" map.

The periodic orbit with period m gives the set of m points on the surface of section

$$(\psi, j)_{l,k} = (l/m, k/m)\ ; \qquad (l, k = 0, 1, \ldots, m-1) \tag{6–5}$$

which is mapped on itself by the transformation (6–4):

$$(\psi, j)_{l,k} \to (\psi, j)_{l+k,k}\ . \tag{6–6}$$

For $k = m$ this gives a fixed point. An orbit with the irrational winding number $\nu(I)$ makes, with time, a KAM circle on the surface of section. The KAM circles break up at some critical nonlinearity K, which is different for different circles. Green[12] related the breakup of the KAM circles to the stability of the neighboring periodic orbits. Let n/m, n'/m' be two rational numbers, so that

$$n/m < \nu < n'/m' \tag{6–7}$$

and $nm' - mn' = \pm 1$. The rationals are neighboring; if not, the interval can be divided into subintervals whose ends are neighboring rational numbers. According to the Poincaré-Birkoff theorem, for each rational winding number there are at least two periodic orbits, one linearly stable (elliptic) and the other linearly unstable (hyperbolic). Thus there are elliptic periodic orbits with winding numbers n/m, n'/m'. The stability of the KAM circle with winding number ν depends on the residues of these periodic orbits. The residue of the periodic orbit is defined as

$$R\left(\frac{n}{m}\right) = \frac{1}{2}\left(1 - \frac{1}{2}TrM^m\right). \tag{6–8}$$

Here M^m is the product of the m matrices, each describing the linearized transformations (6–6) of the periodic orbit:

$$\left(\frac{l}{m}+\delta\psi;\frac{k}{m}+\delta j\right)\rightarrow\left(\frac{l+k}{m}+\overline{\delta\psi};\frac{k}{m}+\overline{\delta j}\right).$$

For the standard map all M are equal:

$$\begin{pmatrix}\overline{\delta\psi}\\ \overline{\delta j}\end{pmatrix}=M\begin{pmatrix}\delta\psi\\ \delta j\end{pmatrix};\qquad M=\begin{pmatrix}1 & 1\\ -K & 1-K\end{pmatrix}. \tag{6–9}$$

According to Greene[12], if both residues $R(n/m)$ and $R(n'/m')$ are small (usually this means $< 1/4$), the KAM circles in the interval (6–7) are not broken. If they are large, there are no KAM circles. The KAM circle that breaks up last in the interval $(n/m\ ,n'/m')$, $m'>m$ has the "noble" winding number

$$\nu_{\text{noble}}=\frac{n+n'\gamma}{m+m'\gamma}\ ,\qquad \gamma=\frac{1}{2}(1+\sqrt{5})\ .$$

For the model (6–9) the residue can be calculated explicitly. First, note that, for stability of the periodic orbit with period m, the frequency ω of the small oscillations of the orbit around the periodic orbit (the Floquet parameters)

$$M^m\begin{pmatrix}\delta\psi\\ \delta j\end{pmatrix}=e^{\pm 2\pi i\omega}\begin{pmatrix}\delta\psi\\ \delta j\end{pmatrix}$$

has to be real. The trace of the matrix M^m is

$$\frac{1}{2}TrM^m=\cos(2\pi\omega)$$

and the residue criterion takes the form

$$R=\sin^2(\pi\omega)<1/4\ . \tag{6–10}$$

The eigenvalues of the matrix M are

$$\lambda_\pm=e^{\pm i\alpha},\qquad \tan\alpha=\frac{\sqrt{K-K^2/4}}{(1-K/2)},$$

so that

$$\frac{1}{2}TrM^m=\frac{1}{2}({\lambda_+}^m+{\lambda_-}^m)=\cos(m\alpha)\ .$$

According to Eq. (6–10), there are KAM circles if

$$2\pi\omega=m\alpha<\pi/3\ .$$

For small K, this gives

$$m\sqrt{K} < \pi/3 \,. \tag{6–11}$$

Let us compare this result with Chirikov's overlap criterion. With the Fourier expansion

$$\delta(\sigma - n) = \sum_k e^{2\pi i \sigma k}$$

the Hamiltonian (6–1) takes the form

$$H(I, \phi, s/R) = (\nu'/2)(I - I_0)^2 - V_0 \sum \cos[m\phi - k(s/R)] \,.$$

There are resonances with tunes

$$m\nu_k = k \,, \qquad \nu_k = \frac{d\phi}{d(s/R)} = \nu'(I_k - I_0) \tag{6–12}$$

for all integer k. Each resonance is described by the resonance Hamiltonian:

$$H(j, \psi, s/R) = (m\nu'/2)j^2 - kj - mV_0 \cos\psi$$

where $\psi = m\phi - 2\pi k\sigma$, $j = (I - I_0)$. The frequency of the small phase oscillations is

$$\Omega^2 = m^2\nu' V_0 = K/4\pi^2.$$

The width of the resonance is

$$\Delta j = \pm\sqrt{4V_0/\nu'} \,, \qquad \Delta\nu = 4\sqrt{\nu' V_0} = 4\Omega/m \,.$$

The distance between resonance (6–12), $\delta\nu - l/m$, and the overlap criterion gives

$$m\sqrt{\nu' V_0} < 1/4 \,, \qquad \text{or} \qquad \sqrt{K} < \pi/2 \,. \tag{6–13}$$

The numeric simulations with the standard model ($m = 1$) give the critical parameter $K_{cr} = 0.97163$. Thus the residue criterion (6–11) gives better numerical results than the overlap criterion (6–13).

As mentioned above, if the nonlinearity parameter K increases, the periodic trajectory becomes unstable and exhibits a period-doubling bifurcation. We can illustrate this by mapping (6–2). For $K > 4$ the orbit with period 1 is unstable. The stable orbit with period 2 appears if $K > 4$ with

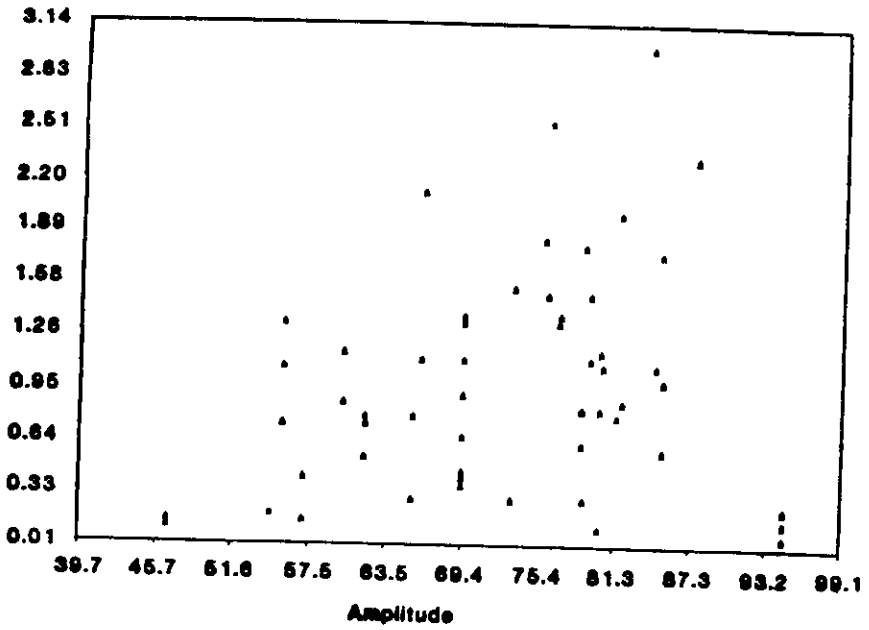
$$j_p = (K/4\pi)\sin\pi j_p \,, \qquad X_p = -j_p/2 \,,$$

which is stable until $(K/4)\cos\pi j_p < 1$.

We try[36] to apply the residue criterion to estimate the maximum amplitude of the stable motion. For simulations we assume that resonances are driven by random multipoles in the dipole magnets. The rms magnitudes of the multipoles were set according to the criteria of linear aperture (3–12). The locations of the resonances on the plane of betatron amplitudes were found from the resonance condition (4–3), where tune included the tune shift (3–6), (3–9), (3–11) given by the random multipoles b_k, $(k = 2, \ldots, 12)$. The SSC parameters $< \beta >= 300$ m, $N_{\rm tot} = 3840, \nu_x{}^0 = 78.265, \nu_y{}^0 = 78.280$ were used for the simulations.

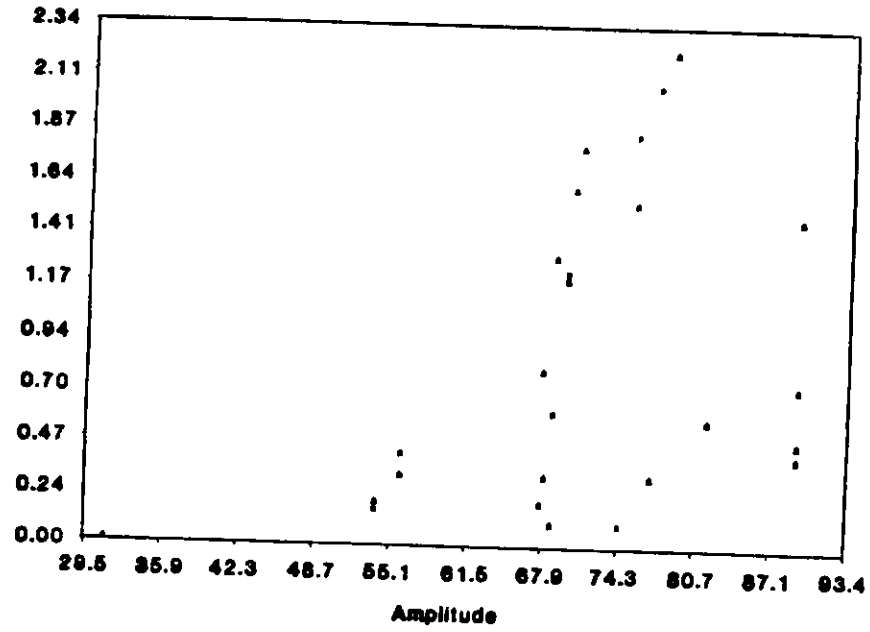
The frequencies of the phase oscillations were calculated according to Eq. (4–10). The parameter of the residue criterion (6–11), understood as $6m\Omega < 1$ with $m = \sqrt{m_x^2 + m_y^2}$, was plotted for each resonance vs. amplitude $\sqrt{A_x^2 + A_y^2}$. The results, shown in Fig. 8, indicate that $6m\Omega < 1$ for amplitudes < 55 rms beam size of about 6 mm. If correlation between random multipoles b_n was taken into account by imposing the restraint on the total field variation $\Delta B/B < 10^{-4}$ at 1 cm, then the residue criterion was satisfied up to amplitudes 67 rms beam size or 7.8 mm.

The calculations with $b_n, n = 2, \ldots, 8$, from Table 4–3.1 of the CDR[1], shown in Fig. 8c, predict stability up to 95 rms amplitudes or 11 mm.

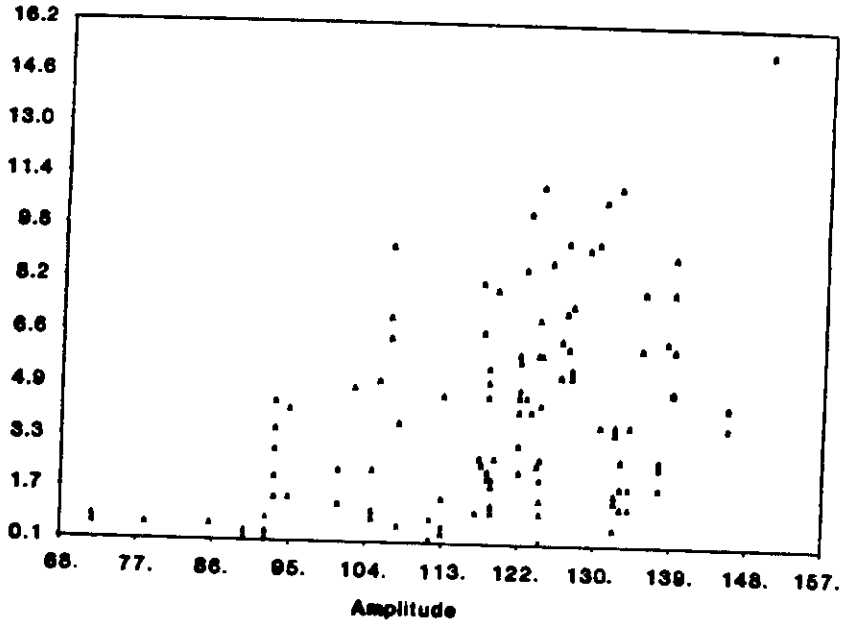


a: for b_n from the Ref. 36 with uncorrelated random multipoles.

Figure 8 The residue criterion parameter $6m\Omega$ vs amplitude of betatron oscillations. The KAM circles break at $6m\Omega = 1$.



b: The same with correlations



c: For the multipoles from CDR[1], Table 4–3.1.

Problem

Consider the stability of the closed orbit, Eq. (1–4), taking into account only two chromatic sextupole families. Assuming that these are set to cancel the linear chromaticity, what is the critical $(\Delta p/p)$ at which the closed orbit bifurcates?

7. CROSSING OF A RESONANCE; SYNCHRO-BETATRON OSCILLATIONS

So far we have considered the motion of an on-momentum particle. Slow oscillations of the momentum with synchrotron frequency Q_s give small oscillations of the betatron tune and Twiss parameters with the same frequency. This is an example of a general situation of slow variations of a parameter. We start with the Hamiltonian (4–6),

$$H(j,\psi,s/R) = H_0 - kj + U_0\cos\psi\ , \tag{7–1}$$

which describes the resonance $m\nu = k$ in the canonical variables

$$\psi = m\phi - k(s/R)\ , \qquad j = I/m$$

and assume the simple form for H_0

$$H_0 = m\nu_0 j + (\nu' j^2/2)\ ; \qquad \nu' = m^2(\partial/\partial I) = \text{constant}\ .$$

The tune ν is $m\nu(j) = d\psi/d(s/R) = m\nu_0 - k + \nu' j$ but ν_0 varies in time with amplitude $\delta\nu$:

$$\nu_0 = \overline{\nu_0} + \delta\nu\cdot\sin(Q_s\cdot s/R)\ . \tag{7–2}$$

The Hamiltonian takes the form

$$H = (\nu'/2)(j - j_r)^2 + U_0\cos\psi\ . \tag{7–3}$$

The width of the resonance is, as usual, $\Delta j = \pm 2\sqrt{U_0/\nu'}$ but the position of the resonance oscillates with frequency Q_s:

$$j_r(s) = \bar{j} - (m\delta\nu/\nu')\sin(Q_s\cdot s/R)\ , \qquad \bar{j} = (k - \bar{\nu}_0 m)/\nu'. \tag{7–4}$$

A particle with amplitude j is in the resonance if $|j - \bar{j}| < \delta\nu/\nu'$. What happens to the particle depends on the parameter

$$v = (m\delta\nu Q_s/\Omega^2) = j_r'/U_0 \tag{7–5}$$

where $\Omega^2 = \nu' U_0$. For the fast crossings of the resonance $v \gg 1$ the system can not follow the change of the tune:

$$\frac{dj}{d(s/R)} \sim U_0 \ll \frac{dj_r}{d(s/R)} \sim \left(\frac{\delta\nu}{\nu'}\right)\cdot Q_s.$$

In this case the Hamiltonian equations

$$\frac{dj}{d(s/R)} = U_0 \sin\psi\ , \qquad \frac{d\psi}{d(s/R)} = \nu'(j - j_r) \tag{7–6}$$

give

$$\psi'' \simeq -\nu' j_r'.$$

If $\delta\nu \gg Q_s$, then j_r' can be considered constant, $j_r' = j_r'(s_r)$ where s_r is the time s at crossing. In this case

$$\psi(s) = \psi_r - (\nu' j_r'/2)[(s/R) - (s_r/R)]^2$$

and the gain of j for a half period is given by (7–6) and (7–7) as

$$j = j_r + U_0\sqrt{(2\pi/\nu' j_r')}\sin(\psi_r + \pi/4)\ . \tag{7–7}$$

This depends on the phase $\psi_r = \psi(s_r)$. If the phases of successive crossings of the resonance are random, then Eq. (7–7) gives the diffusion with the rate

$$\frac{d}{d(s/R)} < \Delta\nu^2 >= \Omega^2 \frac{Q_s}{v}\ , \qquad v \gg 1\ . \tag{7–8}$$

For a slow crossing $v \ll 1$ the system can follow adiabatically the variation of the tune, i.e. the particle can be trapped in the resonance. The total variation of the Hamiltonian (7–3) is

$$\frac{dH}{d(s/R)} = -\nu'(j - j_r) j_r' = -\psi' j_r'.$$

For the slow crossing $j_r' \sim$ const, so that

$$H + j_r'\psi = \text{constant}\ . \tag{7–9}$$

With (7–6) this becomes

$$(\psi'^2/2\nu') + U_0\cos\psi + j_r'\psi = U_0\cos\psi_r + j_r'\psi_r'. \tag{7–10}$$

The motion described by Eq. (7–10) depends on the initial conditions: the particle can be trapped in the potential V_{ef} (see Fig. 9),

$$V_{ef} = U_0\cos\psi + j_r'\psi\ , \tag{7–11}$$

or crosses resonance $\psi = \psi_r$. In the latter case, the variation of the tune $\Delta\nu = \nu'\Delta j$ per crossing is gained mostly by the interval of ψ near the point of reflection ψ_r:

$$\Delta\nu = 2\Omega^2 \int_{\psi_r}^{\infty} d\psi \sin\psi / \sqrt{2\nu'[V_{ef}(\psi_r) - V_{ef}(\psi)]}\ , \quad \Delta\nu \sim -2\Omega v \ln\frac{1}{(v + \psi_r)}\ .$$

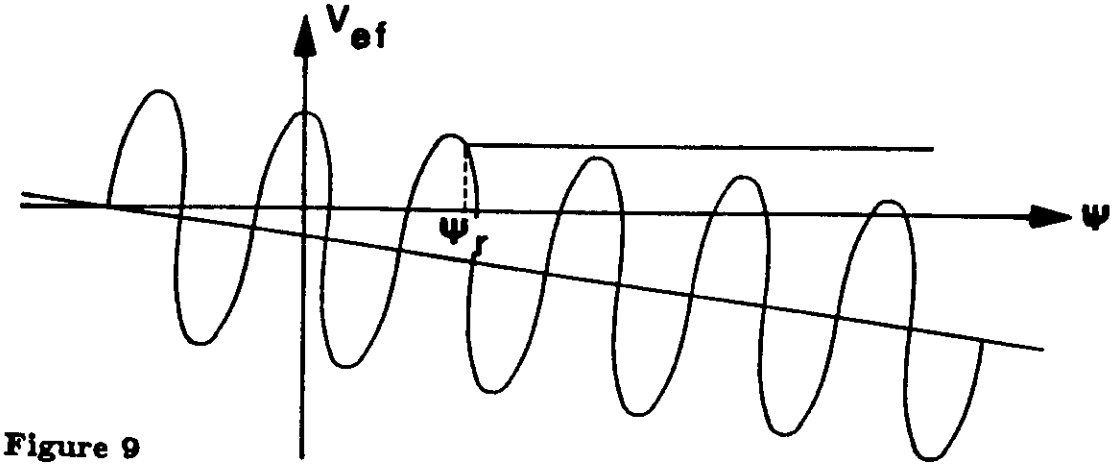


Figure 9
The effective potential for slow crossing of the resonances.

The phase of the crossing is limited in the narrow interval:

$$-v < \psi_r < -v + \sqrt{4\pi v}\,.$$

The diffusion rate is given by

$$\frac{\partial}{\partial(s/R)} < (\Delta\nu)^2 >= 2Q_s\Omega^2 v^2 \ln^2\left(\frac{1}{v}\right) \tag{7-12}$$

and the numeric coefficient is taken from the numeric results.[28] A major question remains: whether the phases ψ_r of the successive crossings are indeed random to give the diffusion. To answer this, we rewrite the Hamiltonian (7–1) in the new canonical variables (X, J), where

$$X = \psi + \lambda\cos[Q_s(s/R)]$$

and the parameter λ is

$$\lambda = m\delta\nu/Q_s. \tag{7-13}$$

The Hamiltonian takes the form

$$H(j, X, s/R) = (\nu'/2)(j - \bar{j})^2 + U_0\cos[X - \lambda\cos(Q_s s/R)]$$

with $\bar{j} = \text{const}$, given by Eq. (7–4). With the expansion

$$e^{i\lambda\cos\alpha} = \sum i^n J_n(\lambda)e^{-in\alpha}$$

over the Bessel functions, the Hamiltonian becomes $\alpha = Q_s(s/R)$:

$$H = \nu'(j - \bar{j})^2/2 + U_0\sum J_n(\lambda)\cos(X - n\alpha + n\pi/2)\,. \tag{7-14}$$

The Hamiltonian (7–14) describes the sideband resonances

$$\nu_n = \nu'(j_n - \bar{j}) = nQ_s\,, \qquad n = \text{integers}\,,$$

around the primary resonance $m\nu_0 = k$. The synchro-betatron sideband resonances have a width $\Delta\nu = 4\Omega_\lambda/m$, where the frequency of small oscillations is

$$\Omega_\lambda = \sqrt{|\nu' U_0 J_n(\lambda)|}\,.$$

For $n \sim \lambda$ the width of all resonances is about the same, $\Omega_\lambda \sim \Omega(\pi\lambda)^{-1/4}$, and goes down exponentially for $n > \lambda$. Therefore, essential resonances are within the band: $\nu_n - k/m < \pm\delta\nu$, with tune separation Q_s/m. According to the Chirikov criterion, the stochasticity emerges if the resonances overlap,

$$K_{\rm ch}{}^2 = (40\Omega^2/Q_s{}^2\sqrt{\pi\lambda}) > 1\,. \qquad (7\text{–}15)$$

The numeric factor is corrected according to ref. 29. The residue criterion $m\Omega_\lambda > 1/6$ gives

$$K_{\rm res}{}^2 = (36m^2\Omega^2/\sqrt{\pi\lambda}) > 1\,. \qquad (7\text{–}16)$$

For small crossing rates (7–5) $v \ll 1$ the criteria (7–15) and (7–16) are fulfilled and there is diffusion (7–12), if only the particle is not trapped in the resonance. For the fast crossing $v \gg 1$, the stochasticity criteria are not necessarily satisfied, and diffusion does not always take place.

For the SSC the modulation of the tune[30] is of the order of the synchrotron tune $Q_s = 2\cdot 10^{-3}$. In this case, the number of sideband resonances (7–13) is equal to the order of the primary resonance. The criteria (7–15) and (7–16) give very different limits on the width of the primary resonance $\Delta\nu_s = 4\Omega/m$:

$$\begin{aligned} &\Delta\nu_s < (Q_s/m)(0.16\pi m)^{1/4} \quad \text{(Chirikov)} \\ \text{and}\quad &\Delta\nu_s < (2/3m^2)(\pi m)^{1/4} \quad \text{(Greene and MacKay).} \end{aligned} \qquad (7\text{–}17)$$

Let us take the residue criterion, and suppose that there is stochasticity $6m\Omega > 1$. The crossing rate (7–5) for $\delta\nu \sim Q_s$ is still small, $v \lesssim 1.5\cdot 10^{-4}\, m^3 \ll 1$, if the order of the primary resonance $m < 19$. This is a reasonable assumption within the linear aperture. The diffusion rate (7–12) in this case gives after 10^8 revolutions

$$<(\Delta\nu)^2> \gtrsim 10^{-4} m^4 \ln^2(1/v).$$

For $m \sim 10$ the width must be much larger than $\Delta\nu \sim 2\cdot 10^{-2}$ to stay in the linear aperature. Thus condition (7–17) is necessary to achieve the luminosity lifetime at the SSC. Fortunately, it is not as tough as the residue criterion applied to the primary resonances, and therefore the synchro-betatron resonances do not cause additional problems.

8. MODULATION DIFFUSION

For a nonlinear resonance, $m_x\nu_x + m_y\nu_y = k$, in the resonance approximation the variables j, ψ (4–5) describe the nonlinear oscillations in the direction transverse to the resonance line. The amplitude B of another degree of freedom remains constant, $d\phi/d(s/R) = m_y\nu_x - m_x\nu_y$. [To avoid confusion we should mention that the directions of the resonance lines on the plane of the amplitude (j, B) and on the tune diagram (ν_x, ν_y) are generally not the same.]

The nonresonance terms of the perturbation give coupling between these two degrees of freedom:

$$dB/d(s/R) = \sum in_2 V(n_1 n_2 n_3) e^{i(n_1\psi + n_2\phi) - in_3 s/R} . \qquad (8\text{–}1)$$

In particular, if the motion (j, ψ) becomes stochastic, the right side of Eq. (8–1) is equivalent to a random external force, which generates diffusion along the stochastic layer, called modulation diffusion. In ref. 14 a simple model of modulation diffusion is considered. Suppose that ψ is the phase of a motion that can be described by a map with period $T = 2\pi/Q$ and a parameter K. If the motion is stochastic (the parameter $K^2 > 1$), it generates a stochastic layer of width $\Delta\nu$.

The amplitude of another degree of freedom varies because of the coupling:

$$dB/d(s/R) = V_0 \sin[\psi - \omega(s/R)] . \qquad (8\text{–}2)$$

In Eq. (8–1), $\omega = d\phi/d(s/R) - n_3$, the diffusion $< (\Delta B)^2 >$ depends on the decoupling of the phase correlations:

$$R = < e^{i\psi(s)} \cdot e^{-i\psi(s')} > .$$

For large $|s - s'|$ Zaslavsky[31] found for the standard model

$$R \simeq e^{-\lambda|s-s'|} , \qquad \lambda = \frac{1}{T} \ln K .$$

The final expression for the diffusion rate is

$$\begin{aligned} \frac{d}{d(s/R)} < (\Delta B)^2 > &= \frac{\pi {V_0}^2}{2\Delta\nu} , && |\omega| < \Delta\nu \\ &= \left(\frac{\pi {V_0}^2 \ln K}{\Delta\nu^2 T} \right) e^{-(2\pi/\Delta\nu)(|\omega| - \Delta\nu)} , && |\omega| > \Delta\nu . \end{aligned} \qquad (8\text{–}3)$$

The diffusion rate is exponentially small for particles with frequencies larger than the width of the stochastic layer. For more details see Vivaldi.[16]

9. NOISE AND NOISE–RESONANCE INTERACTION

Noise in the collider causes diffusion and loss of particles. It has several sources: intrabeam scattering, synchrotron radiation, residual gas, noise from the rf systems, feedback systems, and power supply, Schottky noise of the beam-beam interaction, etc. The estimation gives an emittance growth rate[1] of order of

$$d\varepsilon/dt \sim (10^{-15} + 10^{-17}) \text{ m/sec}, \qquad \varepsilon = \sigma^2/\beta . \tag{9–1}$$

If the amplitude dependence of the tune is of the order of $\partial\nu/\partial\varepsilon \sim 10^4 \text{ m}^{-1}$, it gives a tune shift

$$\Delta\nu \sim \frac{\partial\nu}{\partial\varepsilon} \cdot \frac{d\varepsilon}{dt} \cdot t \sim 10^{-6}$$

for a luminosity lifetime of $t = 1$ day. The diffusion rate can be enhanced if a particle driven by the noise crosses nonlinear resonances. This effect has been considered by Hereward,[32] Neuffer and Ruggiero,[33] and others. We use here the kinetic equation method.[2]

The Hamiltonian $H(\phi, \varepsilon, s/R)$ of a nonlinear system with a random external force $f(s)$ takes the form

$$H = H_0 + V + (R/pc) f(s) X \tag{9–2}$$

where $H_0 = \nu\varepsilon$, V is given by Eq. (2–3), and $X = \sqrt{2\varepsilon\beta}\cos\phi$ is the betatron coordinate of the particle (the one-dimentional case is considered for simplicity). The closed-orbit term is omitted here as unimportant. The general expression for the distribution function F, Eqs. (2–5) and (2–6), can be rewritten as the integral equation

$$F(\alpha, \alpha^+, s) = F_0(\alpha, \alpha^+) - i\int_0^s \frac{ds'}{R}\left[\hat{\mathcal{H}}(s') + \hat{g}(s') f(s')\right] \cdot F(s') \tag{9–3}$$

where the operator $\hat{g}$ corresponds to the last term in the right side of Eq. (9–2):

$$\hat{g}(s) = \frac{R}{pc}\sqrt{\frac{\beta}{2}}\left[e^{-i\mu(s)}\frac{\partial}{\partial\alpha} - e^{i\mu}\frac{\partial}{\partial\alpha^+}\right]$$

and $\mu(s)$ is the phase advance. In the simple case of white noise, the force $f(s)$ is given by the averages

$$< f(s) >= 0 , \qquad < f(s) f(s') >= R d(s) \delta(s - s') . \tag{9–4}$$

Averaging Eq. (9–3) gives

$$< F(s) >= F_0 - i\int_0^s \frac{ds'}{R}\hat{\mathcal{H}}(s') < F(s') > - \\ i\int_0^s \frac{ds'}{R}\hat{g}(s') < f(s')F(s') > \ . \tag{9-5}$$

According to Eq. (9–3), $F(s)$ depends on $f(s')$ with $s > s'$. Therefore, for $s > s'$,

$$< f(s)F(s') >= 0$$

and

$$< f(s)f(s')F(s') >=< f(s)f(s') >< F(s') > \ . \tag{9-6}$$

The first momentum of Eq. (9–3), for this reason, becomes

$$< f(s)F(s) >= \frac{-i}{2}\hat{g}(s)d(s) < F(s) > \ . \tag{9-7}$$

Eqs. (9–5) and (9–7) give finally the Fokker–Plank equation:

$$\frac{\partial < F >}{\partial(s/R)} = -i\hat{\mathcal{H}} < F > -\frac{1}{2}d(s)\hat{g}^2 < F(s) > \ .$$

Using Poisson brackets this then becomes (we drop off $< >$)

$$\frac{\partial F}{\partial(s/R)} = i\{H, F\}_{\alpha\alpha^+} - \frac{d(s)}{2}\{g\{gF\}\}_{\alpha\alpha^+} \tag{9-8}$$

with

$$g = (R/pc)\sqrt{\beta/2}\ e^{-i\mu(s)}\alpha^+ + c.c.$$

In the angle-section variables (ϕ, ϵ) it takes the form

$$\frac{\partial F}{\partial(s/R)} = \{H, F\}_{\phi,\epsilon} + \frac{d(s)}{2}\{g\{gF\}\} \tag{9-9}$$

where $H = \nu\epsilon + V$ is given by Eq. (2–3), and

$$g(s) = (R/pc)\sqrt{2\beta\epsilon}\cos\phi \ . \tag{9-10}$$

The explicit form of Eq. (9–9) is

$$\frac{\partial F}{\partial(s/R)} - \{H_{ef}, F\} \\ = \frac{d(s)}{2}\left[\left(\frac{\partial g}{\partial\phi}\right)^2\left(\frac{\partial^2 F}{\partial\epsilon^2}\right) + \left(\frac{\partial g}{\partial\epsilon}\right)^2\left(\frac{\partial^2 F}{\partial\phi^2}\right) - 2\left(\frac{\partial g}{\partial\phi}\right)\left(\frac{\partial g}{\partial\epsilon}\right)\left(\frac{\partial^2 F}{\partial\epsilon\partial\phi}\right)\right] . \tag{9-11}$$

The right side describes the diffusion with the diffusion rate in ε

$$\frac{d\varepsilon}{d(s/R)} = 2D\ , \qquad D = \left(\frac{\overline{\beta d}}{2}\right)\left(\frac{R}{pc}\right)^2 \tag{9–12}$$

which is generated by the noise. The left side of Eq. (9–11) corresponds to the motion with the effective Hamiltonian H_{ef}, such that

$$\frac{\partial H_{ef}}{\partial \phi} = \frac{\partial H}{\partial \phi} + D\ , \qquad \frac{\partial H_{ef}}{\partial \varepsilon} = \frac{\partial H}{\partial \varepsilon} + \left(\frac{D}{4\varepsilon}\right)\sin 2\phi\ . \tag{9–13}$$

For a particle close to resonance $m\nu = k$, it is convenient to use variables $\psi = m\phi - k$, $J = \varepsilon/m$. After averaging out a small oscillation term, H_{ef} takes the form

$$H_{ef}(\psi, J, s/R) = \frac{\nu'(J - J_r)^2}{2} + U_0 \cos\psi + \left(\frac{D}{m}\right)\psi$$

where $$\nu' = m^2\left(\frac{\partial \nu}{\partial \varepsilon}\right)\ , \qquad J_r = \frac{m\nu - k}{\nu'}\ . \tag{9–14}$$

The Hamiltonian (9–14) has the same structure as the Hamiltonian (7–9) which describes crossing of a resonance. The rate of crossing is $v = \nu' D/m\Omega^2$ where $\Omega = \sqrt{\nu' U_0}$ is the frequency of the phase oscillations of the resonance. For the diffusion rate (9–1) and for reasonable values for the SSC of ν', m, and Ω, the rate $v \ll 1$. Successive crossings of the nonlinear resonances within the linear aperture give an additional diffusion rate (7–12), with $Q_s = (4\pi D/\Delta\nu)(\partial\nu/\partial\varepsilon)$, which depends on the average distance between resonances $\Delta\nu$, including synchro-betatron resonances, and is very small.

The main effect of the noise, therefore, is dragging new particles onto the stochastic layers of the nonlinear resonances, where they can drift away because of Arnold's or modulation diffusion.

10. CONCLUSION

The first and most important step in studying the beam stability of the SSC is the introduction of the linear aperture. Within the linear aperture, there is a hierarchy of generation of resonances, that is, the width of the resonances driven by a nonlinearity of order b^n decreases as $\lambda^{n/2}$, where λ is the parameter of the perturbation theory (1–18). For $\lambda \ll 1$ the number of resonances of higher generations increases more slowly than their width. Therefore, the onset of stochasticity depends on the density and width of the first-generation resonances, driven by the multipoles b_k. These can be calculated explicitly. The residue criterion gives the stability of the orbits up

to amplitudes of 55 to 60 rms beam sizes (or 6.7 to 7.3 mm), which is in good agreement with tracking results[1] (about 8 mm). The synchro-betatron resonances for such amplitudes still do not overlap, and they also increase the total number of resonances of this order of magnitude. Thus they do not change the size of the linear aperture, given above. This should be considered, of course, as a crude approximation, being dependent on the particular set of random multipoles. The Arnold diffusion gives a small effect for the linear aperture.

On the other hand, all effects increase very rapidly with the amplitudes, giving for $\lambda_k > 1$ a diffusion rate too high to confine particles for the 10^8 revolutions needed for the luminosity lifetime. This means that the concept of the dynamic aperture[1] can be useful only for small numbers of turns.

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